DESIGN OF MULTIVARIABLE POF CONTROLLER FOR
SMART COMPOSITE BEAM USING EMBEDDED SHEAR
SENSORS AND ACTUATORS

T.C. MANJUNATH, B. BANDYOPADHYAY

Interdisciplinary Programme in Systems & Control Engineering,
101 B, ACRE Building, Indian Institute of Technology Bombay, Powai,
Mumbai - 400076, Maharashtra, India.
Email : tcmanju@sc.iitb.ac.in ; bijnan@ee.iitb.ac.in
URL : http://www.sc.iitb.ac.in/~tcmanju ; http://www.sc.iitb.ac.in/~bijnan
Phone : +91 22 25767884 ; +91 22 25767889 ; Fax : +91 22 25720057

Abstract: In this paper, the modelling and design of a multivariable controller for a smart thick composite
cantilever beam with embedded shear sensors and actuators is investigated. The first 3 dominant vibratory
modes are retained in the modelling of the composite beam. The beam is divided into 8 finite elements and shear
piezoelectric patches are embedded into the master structure. These shear piezoelectric patches serve as sensors
and actuators at 2 finite element locations to obtain a multivariable system with 2 inputs and 2 outputs. The
beam is subjected to an external disturbance at the free end. The vibrations are damped out quickly when the
POF controller is put in a feedback loop with the beam. The effect of placing the non-collocated sensor / actuator
pairs at 2 finite element locations in between the two beam layers is observed and the conclusions are drawn.
The closed loop responses with the output feedback gain were found to be satisfactory. Shear and axial
displacements, neglected in the classical Euler-Bernoulli beam theory are considered in this research to produce
an accurate beam model.

Keywords: Smart structure, Timoshenko, POF, FEM, State space model, vibration control, LMI.

1. INTRODUCTION

Smart materials and smart structures, often called as
the intelligent structures, form a new rapidly
growing interdisciplinary technology in the modern
day world embracing the fields of materials,
structures, sensors and actuators, information and
signal processing, electronics and control [Gandhi
and Thompson, 1992]. A smart structure
incorporates distributed actuators and sensors and
has the data processing and power conditioning
capabilities. Also, it has the capability to respond to
a changing external environment (such as loads and
shape changes) as well as to a changing internal
environment (such as damage or failure).

Smart structures involve the synergism of intelligent
materials with embedded or surface mounted sensors
whose information is collected, processed and
controlled by a sophisticated controller, which
controls the actuator to perform the corrective
action. Recent advances in smart structure
technology provide [Chopra, 2002] a means for
integrating sensors and actuators into the structure
and make them self-adapting, self-controlling, and
intelligent in various types of mechanical, flexible
and rigid engineering structures. These include
aerospace applications, civil engineering
applications, robotics, bio-technology, MEMS and
NEMS. The need for such intelligent structures
called smart structures [Culshaw, 1992] arises
because of their high performance in numerous
structural applications.

Such intelligent structures incorporate smart
materials called actuators and sensors (based on
Piezoelectrics, MR Fluids, Piezo-ceramics, ER
Fluids, SMA, PVDF, Optical fibres, etc.) that are
embedded into the structure. They have structural
functionality with highly integrated control logic,
signal conditioning, and power amplification
electronics. These materials can be used to generate
a secondary vibrational response in a mechanical
system. This secondary response has the potential to
reduce the overall response of the system by the
destructive interference with the original response of
the system, caused by the primary source of
vibration [Herman, 1994].

Piezoelectric materials [Rao and Sunar, 1994] are
used in our research work as embedded shear
sensors and actuators to suppress the structural
vibrations. Considerable interest is focused on the
modelling, control and implementation of smart
structures using the Euler-Bernoulli beam theory and
the Timoshenko beam theory with integrated
piezoelectric layers in the recent past. The
assumption made in the Euler Bernoulli beam theory
is that plane cross sections of the beam remain plane
and normal to the neutral axis after deformation.
Since the shear forces and axial displacements are neglected in the Euler-Bernoulli theory, slightly inaccurate results may be obtained.

The Timoshenko Beam Theory is used to overcome the drawbacks of the Euler-Bernoulli beam theory by considering the effect of shear and axial displacements. In the Timoshenko beam theory, the plane cross sections of the beam remain plane and rotate about the same neutral axis as the Euler-Bernoulli model, but do not remain normal to the neutral axis after deformation. The deviation from normality is produced by a transverse shear that is assumed to be constant over the cross section. The total slope of the beam in this model consists of two parts, one due to bending θ and the other due to shear γ. Thus, the Timoshenko beam model is superior to Euler-Bernoulli beam model in accurately predicting the beam response. Thus, this model corrects the classical beam model with first-order shear deformation effects. The following few paragraphs give a deep insight into the research work done on the intelligent structures using the 2 types of theories so far.


A passivity-based control for smart structures was designed by Gosavi and Kelkar [2004]. A self tuning active vibration control scheme in flexible beam structures was carried out by Tokhi [1994]. Active control of adaptive laminated structures with bonded piezoelectric sensors and actuators was investigated by Moita et al. [2004]. Ullrich et al. [2002] devised a optimal LQG control scheme to suppress the vibrations of a cantilever beam. Finite element simulation of smart structures using an optimal output feedback controller for vibration and noise control was performed by Young et al. [1999]. Work on vibration suppression of flexible beams with bonded piezo-transducers using wave-absorbing controllers was done by Vukovich and Koma [2000]. Aldraihem et al. [1997] have developed a laminated beam model using two theories: namely, the Euler-Bernoulli beam theory and the Timoshenko beam theory. Abramovich [1998] has presented the analytical formulation and closed form solutions for composite beams with piezoelectric actuators, based on the Timoshenko beam theory. He also studied the effects of actuator location and number of patches on the actuator’s performance. The study considered various configurations of the piezo patches and boundary conditions under mechanical and/or electric loads. Using a higher-order shear deformation theory, Chandrashekhara and Varadarajan [1997] presented a finite element model of a composite beam to produce a desired deflection in beams with clamped-free, clamped-clamped and simply supported ends.

Aldraihem and Khdeir [2000] proposed analytical models and exact solutions for beams with shear and extension piezoelectric actuators. The models were based on the Timoshenko beam theory and higher-order beam theory. Exact solutions were obtained by using the state-space approach. Doschner and Enzmann [1998] designed a model-based controller for smart structures. Sun and Zhang [1995] suggested the idea of exploiting the shear mode to create transverse deflection in sandwich structures. Here, he proved that embedded shear actuators offer many advantages over surface mounted extension actuators. Robust multivariable control of a double beam cantilever smart structure was implemented by Robin Scott et al. [2003]. In a more recent work, Zhang and Sun [1996] formulated an analytical model of a sandwich beam with a shear piezoelectric actuator that occupies the entire core. The model derivation was simplified by assuming that the face layers follow the Euler-Bernoulli beam theory, whereas the core layer obeys the Timoshenko beam theory. Furthermore, a closed form solution of the static deflection was presented for a cantilever beam. A new method of modelling and controlling the
shape of composite beams with embedded piezoelectric actuators was proposed by Donthireddy and Chandrashekhara [1996].

A reference method of controlling the vibrations in flexible smart structures was shown by Murali et al. [1995]. Thomas and Abbas [1975] explained some techniques of performing finite element methods for the dynamic analysis of Timoshenko beams. A FEM approach was used by Benjeddou et al. [1999] to model a sandwich beam with shear and extension piezoelectric elements. The finite element model employed the displacement field of Zhang and Sun [1996]. It was shown that the finite element results agree quite well with the analytical results. An improved 2-node Timoshenko beam model incorporating the axial displacement and shear was presented by Kosmataka and Friedman [1993]. The finite element model of Benjeddou’s research team was extended by Raja et al. [2002] to include a vibration control scheme. Azulay and Abramovich [2004] have presented the analytical formulation and closed form solutions for composite beams with piezoelectric actuators. Abramovich and Lishvits [1998] extended their work to the active stiffening of composite beams. Abramovich and Waisman [2002] proposed in closed form solutions by Abramovich [1994]. Further, he and Waisman [2002] extended the finite element model of Benjeddou’s research team was extended by Raja et al. [2002] to include a vibration control scheme. Azulay and Abramovich [2004] have presented the analytical formulation and closed form solutions for composite beams with piezoelectric actuators. Abramovich and Lishvits [1998] extended their work to the active stiffening of composite beams. Abramovich and Waisman [2002] extended their work to the active stiffening of composite beams. Abramovich and Waisman [2002] extended their work to the active stiffening of composite beams. Abramovich and Waisman [2002] extended their work to the active stiffening of composite beams. Abramovich and Waisman [2002] extended their work to the active stiffening of composite beams. Abramovich and Waisman [2002] extended their work to the active stiffening of composite beams. Abramovich and Waisman [2002] extended their work to the active stiffening of composite beams. Abramovich and Waisman [2002] extended their work to the active stiffening of composite beams.

The outline of the paper is as follows. A brief review of the literature on existing beam models was given in the introductory section. Section 2 gives an overview of the modeling technique (sensor / actuator model, finite element model, state space model) for the smart cantilever beam based on the Timoshenko beam theory. A brief review of the periodic output feedback controlling technique is presented in Section 3, followed by the design of the controller for the first three vibratory modes for the MIMO model. The simulation results are presented in Section 5. Conclusions are drawn in Section 6 followed by the appendix, nomenclature, and references.

2. MODELLING OF TIMOSHENKO BEAM

In this Section, a shear deformable (Timoshenko) Finite Element Model [Seshu, 2004] is developed for a laminated beam and its application in active vibration control is investigated [Abramovich, 1998], [Abramovich and Lishvits, 1994], [Abramovich and Waisman, 2002]. An accurate model of the system is obtained when the shear effects and the axial displacement of the beam are considered in modeling the smart structure. A sandwiched beam (piezo-laminated composite beam) shown in figure 1 consists of 3 layers, namely the piezo-patch with the rigid foam sandwiched in between two aluminum beam layers. For shear actuation, rigid foam is introduced as a core along with PZT to obtain an equivalent sandwiched model. The assumption made is that the middle layer is perfectly glued to the carrying structure and the thickness of the adhesive is neglected (thus, neglecting the effect of shear-lag, no slippage or delamination between the core layers during vibrations) as a result of which strong coupling exists between the master structure and the piezo-patches. The beam is divided into 8 finite elements and the actuators are placed at finite element locations 2 and 4, whereas the sensors are placed at finite element locations 6 and 8 respectively. The properties of the beam and those of the piezo patches are given in the tables 1 and 2 respectively.

Table 1: Properties of the beam

<table>
<thead>
<tr>
<th>Parameter (with units)</th>
<th>Symbol</th>
<th>Numerical values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of beam (cm)</td>
<td>L_b</td>
<td>20</td>
</tr>
<tr>
<td>Width (cm)</td>
<td>c</td>
<td>2</td>
</tr>
<tr>
<td>Thickness of the top layer and bottom Al beam layers (mm)</td>
<td>t_b</td>
<td>1</td>
</tr>
<tr>
<td>Young’s modulus (GPa)</td>
<td>E_b</td>
<td>193.06</td>
</tr>
<tr>
<td>Density (kg/m³)</td>
<td>ρ_h</td>
<td>8030</td>
</tr>
<tr>
<td>Damping constants α, β</td>
<td></td>
<td>0.001, 0.0001</td>
</tr>
</tbody>
</table>

Table 2: Properties of the piezoelectric patch

<table>
<thead>
<tr>
<th>Parameter (with units)</th>
<th>Symbol</th>
<th>Numerical values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (cm)</td>
<td>l_p</td>
<td>2.5</td>
</tr>
<tr>
<td>Width (cm)</td>
<td>c</td>
<td>2</td>
</tr>
<tr>
<td>Thickness (mm)</td>
<td>t_a, t_s</td>
<td>1</td>
</tr>
<tr>
<td>Young’s modulus (GPa)</td>
<td>E_p</td>
<td>84.1</td>
</tr>
<tr>
<td>Density (kg/m³)</td>
<td>ρ_p</td>
<td>7900</td>
</tr>
<tr>
<td>Piezo strain constant (m/V)</td>
<td>d_{31}</td>
<td>– 274.8 × 10^{-12}</td>
</tr>
</tbody>
</table>

The perfect bonding of the adhesive between the beam and the sensor / actuator and the bottom and top surfaces of the upper and lower aluminum beam have been assumed to add no mass or stiffness to the sensor-actuator. For the parts without piezoelectrics, the extra space at places where no piezoelectrics are
The bending slope and the additional (axial) direction. The displacement, \( w(x, z) \) is the bending rotation of the element lies along the temperature effects have been neglected. The gain device has been considered negligible and the capacitance between sensor and signal conditioning density and length of the element. The cable theory with constant MI, modulus of elasticity, mass field is based on a first order shear deformation such as honeycomb or rigid foam are highly efficient in producing bending and shear [Sun and Zhang, 1995; Zhang and Sun, 1996].

The poling direction of the piezoelectric patch is along the \( x \) (axial) direction. The displacement field is based on a first order shear deformation theory with constant MI, modulus of elasticity, mass density and length of the element. The cable capacitance between sensor and signal conditioning device has been considered negligible and the temperature effects have been neglected. The gain \( (G) \) of the signal conditioning device is assumed as 100. The longitudinal axis of the sandwiched beam element lies along the \( x \) - axis and the beam is subjected to vibrations in the \( x \) - \( z \) plane.

The beam element is assumed to have 3 structural DOF, \( (w, \theta) \) at each nodal point, \( w \) being the transverse displacement and an axial displacement \( u \) of the nodal point. A bending moment and a transverse shear force act at each nodal point. \( \frac{dw}{dx} \) is the slope of the beam (composed of 2 parts, \( \theta(x) \), the bending slope and the additional shear deformation \( \gamma(x) \)). An additional DOF, called the electrical DOF (sensor voltage) is present when the piezo patches are used on the beam. Since the voltage is constant over the electrode, the number of electrical DOF is one for each element.

### 2.1 Modelling of the Regular Beam Element

The equations of motion of a general piezolaminated composite beam is obtained as follows [Abramovich, 1998; Abramovich and Lishvits, 1994; Abramovich and Waisman, 2002]. The displacements of the beam \( u(x) \) and \( w(x) \) can be written as

\[
u(x, z) = u_0(x) + z \theta(x, t),
\]

\[
w(x, z) = w_0(x),
\]

where \( u_0(x) \) and \( w_0(x) \) are the axial and lateral displacements of the point at the mid-plane assuming that there is incompressibility in the \( z \) direction, and \( \theta(x) \) is the bending rotation of the normal to the mid-plane, namely the rotation of the beam about the \( y \) axis. The total strain vector is the sum of the mechanical strain vector and the actuator induced strain vector.

The strain components of the beam are given as

\[
\varepsilon_x = \frac{\partial u_0}{\partial x} + z \frac{\partial \theta}{\partial x},
\]

\[
\varepsilon_z = 0,
\]

\[
\gamma_{xz} = \frac{\partial u}{\partial x} + \frac{\partial w_0}{\partial x} = \theta + \frac{\partial w_0}{\partial x}.
\]

where \( \varepsilon_x, \varepsilon_z \) are the mechanical normal and transverse shear strain, \( \gamma_{xz} \) being the shear strain induced in the piezoelectric layer. The beam constitutive equation can be written as

\[
\begin{bmatrix}
N_x \\
M_x \\
Q_{xz}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & B_{11} & 0 \\
B_{11} & D_{11} & 0 \\
0 & 0 & A_{55}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u_0}{\partial x} \\
\frac{\partial u}{\partial x} \\
\theta + \frac{\partial w_0}{\partial x}
\end{bmatrix} -
\begin{bmatrix}
E_{11} \\
F_{11} \\
G_{55}
\end{bmatrix},
\]

\[
N_x = \int c \sigma_x z dz,
\]

\[
M_x = \int c \sigma_z z dz,
\]

\[
Q_{xz} = \int c \tau_{xz} dz.
\]

Here, \( \sigma_x = \Theta_{11} \varepsilon_x \) and \( \tau_{xz} = \Theta_{55} \gamma_{xz} \) are the normal and shear stresses respectively and \( c \) is the width of the beam, \( z \) is the depth of the material point measured from the beam reference plane along the vertical axis. \( h \) is the height of the beam plus the piezo-patch, namely the thickness of the total structure, which includes \( t_b, t_a, t_s \) (thickness of beam, thickness of actuator / sensor). \( N_x, M_x, Q_{xz} \) are the internal forces acting on the cross section of the beam. \( A_{11}, B_{11}, D_{11} \) and \( A_{55} \)
are the extensional, bending-extensional, bending and transverse shear stiffness coefficients defined according to the laminate theory as

\[ A_{11} = c \sum_{k=1}^{N} \left( \overrightarrow{Q}_{11} \right)_k \left( z_k - z_{k-1} \right), \]  
\[ B_{11} = c \sum_{k=1}^{N} \left( \overrightarrow{Q}_{11} \right)_k \left( z_k^2 - z_{k-1}^2 \right), \]  
\[ D_{11} = c \sum_{k=1}^{N} \left( \overrightarrow{Q}_{11} \right)_k \left( z_k^3 - z_{k-1}^3 \right), \]  
\[ A_{55} = c K \sum_{k=1}^{N} \left( \overrightarrow{Q}_{55} \right)_k \left( z_k - z_{k-1} \right). \]  

Here, in equations (10) to (13), \( z_k \) is the distance of the \( k^{th} \) layer from the x-axis, \( N \) is the number of layers, \( K \) is the shear correction factor [Cooper, 1966] usually taken equal to \( \frac{5}{6} \) [Ahmed and Osama, 2001] and \( \overrightarrow{Q}_{11}, \overrightarrow{Q}_{55} \) are calculated according to the equations using the material properties of the piezoelectric material as given by

\[ \overrightarrow{Q}_{11} = Q_{11} \cos^4 \lambda + Q_{22} \sin^4 \lambda + 2 \left( Q_{12} + 2 Q_{66} \right) \sin^2 \lambda \cos^2 \lambda, \] \[ \overrightarrow{Q}_{55} = G_{13} \cos^2 \lambda + G_{23} \sin^2 \lambda. \]

The angle \( \lambda \) is the angle between the fiber direction and the longitudinal axis of the beam. The material constants \( Q_{11}, Q_{22}, Q_{12}, Q_{66}, Q_{13} \) and \( Q_{23} \) for foam, aluminum and piezoelectric material were taken from the data handbook [Myer, 2002]. These constants are used to calculate the values of \( A_{11}, B_{11}, D_{11} \) and \( A_{55} \) using equations (10) to (15). \( E_{11}, F_{11} \) and \( G_{55} \) in equation (6) are the actuator induced axial force, bending moment and the shear force respectively, defined as

\[ E_{11} = c \sum_{k=1}^{N} \left( \overrightarrow{Q}_{11} \right)_k V^k(x,t) d_{31}^k, \]  
\[ F_{11} = c \sum_{k=1}^{N} \left( \overrightarrow{Q}_{11} \right)_k V^k(x,t) \left( z_k^a - z_{k-1}^a \right), \]  
\[ G_{55} = c K \sum_{k=1}^{N} \left( \overrightarrow{Q}_{55} \right)_k V^k(x,t) d_{15}^k. \]

Since the piezoelectric layer is poled in the axial direction, \( E_{11} = F_{11} = 0 \). \( V^k(x,t) \) is the applied voltage to the \( k^{th} \) actuator having a thickness of \( \left( z_k^a - z_{k-1}^a \right) \) and \( d_{31}^k, d_{15}^k \) are the piezoelectric constants. \( \overrightarrow{Q}_{11} \) and \( \overrightarrow{Q}_{55} \) are the coefficients of the actuators calculated using the equations (14) and (15). \( N_a \) is the number of actuators, where ‘a’ stands for ‘w.r.t. actuator’. Using the Hamilton’s principle (total strain energy is equal to the sum of the change in the kinetic energy plus the work done due to the external forces), we get

\[ \delta \Pi = \int_{t_1}^{t_2} \left( \delta T - \delta U + \delta W \right) dx dt, \]

where \( T \) is kinetic energy, \( U \) is strain energy, \( W \) is the external work done, \( L \) is the length of the beam element and \( t \) is the time.

The strain energy \( U \) of the beam element is given by

\[ \delta U = N_a \left( \frac{\partial^2 u}{\partial x^2} \right) + M_a \left( \frac{\partial^2 \theta}{\partial x^2} \right) + \sum_{k=1}^{N_a} \left( \theta + \frac{\partial^2 w}{\partial x^2} \right), \]

The kinetic energy \( T \) of the beam element is given by

\[ \delta T = \left( I_1 \ddot{u} + I_2 \dot{\theta} \right) \ddot{u} + I_1 \dot{w} \ddot{w} + \left( I_2 \ddot{u} + I_3 \dot{\theta} \right) \dot{\theta}. \]

Here, in equation (21), \( I_1, I_2 \) and \( I_3 \) are the mass inertias defined as

\[ I_1 = \int_{-h/2}^{h/2} \rho dz, \]  
\[ I_2 = \int_{-h/2}^{h/2} z \rho dz, \]  
\[ I_3 = \int_{-h/2}^{h/2} z^2 \rho dz, \]

where \( \rho \) is the mass density of each layer and \( h \) is the height of the beam including the piezo-patch, namely the thickness of the total structure.

The external work done (i.e., force×displacement) is given as

\[ \delta W = q_0 \delta w, \]

where \( q_0 \) is the transverse distributed load. Substituting the values of strain energy, kinetic energy and external work done from equations (20), (21) and (25) into equation (19), we get the governing equation of motion of a general shaped beam.
non-symmetric piezo-laminated beam with shear deformation and rotary inertia as
\[
\frac{\partial}{\partial x} \left( A_{11} \frac{\partial u}{\partial x} + B_{11} \frac{\partial \theta}{\partial x} + E_{11} \right) = \frac{\partial}{\partial t} \left[ I(\dot{u}) + (I \dot{\theta}) \right],
\]
(26)
\[
\frac{\partial}{\partial x} \left( A_{55} \left( \theta + \frac{\partial w}{\partial x} \right) + G_{55} \right) = \frac{\partial}{\partial t} \left[ I(\dot{w}) + q_{0} \right],
\]
(27)
\[
\frac{\partial}{\partial x} \left( B_{11} \frac{\partial u}{\partial x} + D_{11} \frac{\partial \theta}{\partial x} + F_{11} \right) - A_{55} \left( \theta + \frac{\partial w}{\partial x} \right) - G_{55} = \frac{\partial}{\partial t} \left[ (I \dot{\theta}) + (I \dot{u}) \right],
\]
(28)
which becomes
\[
\frac{\partial}{\partial x} \left( A_{11} \frac{\partial u}{\partial x} + B_{11} \frac{\partial \theta}{\partial x} \right) = 0,
\]
(29)
\[
\frac{\partial}{\partial x} \left( A_{55} \left( \theta + \frac{\partial w}{\partial x} \right) \right) = 0,
\]
(30)
\[
\frac{\partial}{\partial x} \left( B_{11} \frac{\partial u}{\partial x} + D_{11} \frac{\partial \theta}{\partial x} + F_{11} \right) - A_{55} \left( \theta + \frac{\partial w}{\partial x} \right) - G_{55} = 0
\]
(31)
for a static case and with constant properties of the beam.

To facilitate the solution process for the coupled equations in equations (29-31), the beam stiffness
\(A_{55}\) and \(D_{11}\) are assumed to be uniform and constant throughout the beam length [Abramovich, 1998], [Abramovich and Lishvits, 1994], [Abramovich and Waisman, 2002]. Note that the influence of shear-induced strains appears in the above coupled equations of motion for constant properties along the beam. Let
\[
w = a_1 + a_2 x + a_3 x^2 + a_4 x^3,
\]
(32)
\[
\theta = b_1 + b_2 x + b_3 x^2,
\]
(33)
\[
u = c_1 + c_2 x + c_3 x^2
\]
(34)
be the solutions of the equations (29) to (31) where \(a_i, b_j\) and \(c_j\)'s are the unknown coefficients
\((i=1, \ldots, 4)\) and \((j=1, \ldots, 3)\) subject to the boundary conditions
at \(x = 0\) \(w = w_1\) \(\theta = \theta_1\) \(u = u_1\),
at \(x = L\) \(w = w_2\) \(\theta = \theta_2\) \(u = u_2\),
(35)
where \(x\) is the local axial coordinate of the element.

After applying the boundary conditions from equation (35) into equations (32-34), the unknown coefficients \(a_i, b_j\) and \(c_j\)'s can be resolved. Since the axial displacement of a point not on the centerline is a linear function of \(\theta\) as well as of \(u\), the degree of the polynomial used for \(\theta\) must be the same as that used for \(u\). In addition, the shear strain is a linear function of both \(\theta\) and \(\frac{dw}{dz}\).

To ensure compatibility, the degree of the polynomial used for \(w\) must be one order higher than those used for \(u\) and \(\theta\). Therefore, for consistency, the cubic function used for the displacement \(w\) requires that quadratic functions be used for both the axial displacement \(u\) and the cross sectional rotation \(\theta\). Then, substituting the expressions for the unknown coefficients into equations (32-34), we get the expressions for the axial displacement, transverse displacement, and bending rotation in matrix form as
\[
[u] = [N_u] \begin{bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{bmatrix},
\]
(36)
\[
[w] = [N_w] \begin{bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{bmatrix},
\]
(37)
and
\[
[\theta] = [N_\theta] \begin{bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{bmatrix},
\]
(38)
\(N_u, N_w, N_\theta\) are the mode shape functions due to the axial displacement, transverse displacement and bending rotation or the slope, which are defined as
\[
[N_u] = [N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6],
\]
(39)
\[
[N_w] = [N_7 \ N_8 \ N_9 \ N_{10}],
\]
(40)
\[
[N_\theta] = [N_{11} \ N_{12} \ N_{13} \ N_{14}]
\]
(41)
with the elements of the shape function given by
\[
N_1 = 1 - \frac{x}{l},
\]
(42)
\[
N_2 = \frac{6}{(12 \eta - l^2)} x - \frac{6 \gamma}{l(12 \eta - l^2)} x^2
\]
(43)
\[
N_3 = -\frac{6}{(12 \eta - l^2)} x + \frac{6 \gamma}{l(12 \eta - l^2)} x^2,
\]
(44)
\[
N_4 = \frac{x}{l}
\]
(45)
The mass matrix of each regular beam element is given by

\[
[M] = \int_{0}^{l} [N]^{T} [I] [N] \, dx.
\]

where

\[
[I] = \begin{bmatrix}
I_1 & 0 & I_2 \\
0 & I_1 & 0 \\
I_2 & 0 & I_3
\end{bmatrix}
\]

is the inertia matrix and \(I_1, I_2\) and \(I_3\) are given by equations (22) to (24) respectively. The element mass matrix is given by [Moita et al., 2004]

\[
[M] = \begin{bmatrix}
M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\
M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} \\
M_{31} & M_{32} & M_{33} & M_{34} & M_{35} & M_{36} \\
M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46} \\
M_{51} & M_{52} & M_{53} & M_{54} & M_{55} & M_{56} \\
M_{61} & M_{62} & M_{63} & M_{64} & M_{65} & M_{66}
\end{bmatrix}
\]

Here, \([M]\) is a symmetric matrix called the local matrix, namely the mass matrix of the small finite element [Zapfe et al., 1999; Lee 2000, Louis et al., 2002]. The values of the mass matrix coefficients are given in the appendix. The stiffness matrix of the particular regular beam element [Moita et al., 2004] is given by

\[
[K] = \int_{0}^{l} [B]^{T} [D] [B] \, dx,
\]

where \(A\) is the area of the cross section and

\[
[B] = \frac{d[N]}{dx},
\]

\[
[D] = \begin{bmatrix}
A_{11} & B_{11} & 0 \\
B_{11} & D_{11} & 0 \\
0 & 0 & A_{55}
\end{bmatrix},
\]

\[
[K] = \begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66}
\end{bmatrix}
\]

Here, \([K]\) is a symmetric matrix called as the local stiffness matrix [Zapfe et al., 1999; Lee 2000, Louis et al., 2002]. The values of the matrix coefficients are given in the appendix. The mass and stiffness matrices of the regular beam element are obtained using foam as the core between two facing aluminum layers. The mass and stiffness matrices of the piezoelectric beam element are obtained by using a shear piezoelectric patch between the two facing aluminum layers.
2.2 Sensor and Actuator Equations

In this Section, the modelling of the sensor and actuator equations are presented.

2.2.1 Sensor equation

When a force acts upon a piezoelectric material, an electric field is produced [Rao and Sunar, 1994], [Manju and Bijnan, 2004]. This effect, which is called direct piezoelectric effect, is used to calculate the output charge produced by the strain in the structure. The external field produced by the sensor is directly proportional to the strain rate. The charge \( q(t) \) accumulated on the piezoelectric electrodes using the Gauss law is given by

\[
q(t) = \int \limits_A D_3 \, dA,
\]

where \( D_3 \) is the electric displacement in the thickness direction and \( A \) is the area of the electrodes. If the poling is done along the axial direction of the sensors with the electrodes on the upper and lower surfaces, the electric displacement is given by

\[
D_3 = \varphi_{35} \, d_{15} \, \gamma_{xz} = e_{15} \, \gamma_{xz},
\]

where \( e_{15} \) is the piezoelectric constant. On solving equation (67), we get

\[
q(t) = e_{15} \frac{6\eta}{(-12\eta + l^2)} \begin{bmatrix} 0 & 2 & -l & 0 & -2 & -l \end{bmatrix} \dot{\mathbf{q}},
\]

where \( \dot{\mathbf{q}} \) is the time derivative of the modal coordinate vector (strain rate), \( l \) is the length of the piezo patch and \( \mathbf{p}^T \) is a constant vector of size \((1 \times 6)\) for a 2 node element which depends on the type of sensor and its finite element location in the embedded structure and is given by

\[
\mathbf{p}^T \mathbf{q} = \begin{bmatrix} 0 & 2 & -l & 0 & -2 & -l \end{bmatrix} \dot{\mathbf{q}}.
\]

The input voltage to the actuator is \( V^a(t) \) and is given by

\[
V^a(t) = \mathbf{K} \, V^s(t),
\]

\[
V^a(t) = \frac{6 \eta e_{15} c G_c}{(-12 \eta + l^2)} \begin{bmatrix} 0 & 2 & -l & 0 & -2 & -l \end{bmatrix} \dot{\mathbf{q}},
\]

where \( \mathbf{K} \) is the gain of the controller. The sensor output voltage is a function of the second spatial derivative of the mode shape.

2.2.2 Actuator equation

The strain produced in the piezoelectric layer is directly proportional to the electric potential applied to the layer and is given by [Rao and Sunar, 1994], [Manjunath and Bandopadhyay, 2004]

\[
\gamma_{xz} \propto E_f,
\]

where \( \gamma_{xz} \) is the shear strain in the piezoelectric layer, and \( E_f \) is the electric potential applied to the actuator. From the constitutive piezoelectric equation, we get,

\[
\gamma_{xz} = d_{15} \, E_f.
\]

Since the ratio of shear stress to shear strain is the
modulus of rigidity $G$, the shear stress is given by

$$
\tau_{xz} = G \dot{\gamma}_{xz}.
$$

(79)

Substituting the value of $\dot{\gamma}_{xz}$ from equation (78) into equation (79), we get

$$
\tau_{xz} = G d_{1s} E_f
$$

(80)

and

$$
E_f = \frac{V^a(t)}{t_p},
$$

(81)

where $t_p$ is the thickness of the piezoelectric layer.

Thus,

$$
\tau_{xz} = G d_{1s} \frac{V^a(t)}{t_p}.
$$

(82)

Because of the stress and strain, bending moments are induced in the beam at the nodes and the resulting moment $M_a$ acting on the beam is determined by integrating the stress throughout the structure thickness as

$$
M_a = G d_{1s} \int V^a(t) \ h = G d_{1s} K_c p^t \dot{q} \ h,
$$

(83)

where $\ h = \frac{(t_a + t_b)}{2}$ is the distance between the neutral axis of the beam and the piezoelectric layer.

The work done by this moment results in the generation of the control force that is applied by the actuator as

$$
f_{ctrl} = G d_{1s} \ h \int_0^{l_p} N_\theta \ dx \ V^a(t)
$$

(84)

or can be expressed as a scalar product

$$
f_{ctrl} = h^T V^a(t),
$$

(85)

where $h^T$ is a constant vector of size $(6 \times 1)$ for a 2 node element which depends on the type of actuator and its finite element location in the embedded structure and $d_{1s}$ is the piezoelectric strain constant. If any external forces described by the vector $f_{ext}$ are acting then, the total force vector becomes

$$
f' = f_{ext} + f_{ctrl}.
$$

(86)

2.3 Dynamic Equation of the Smart Structure

The dynamic equation of the smart structure is obtained by using both the regular and the piezoelectric beam elements (local matrices) given by equations (60) and (64). The mass and stiffness of the bonding or the adhesive between the master structure and the sensor / actuator pair is neglected. The mass and stiffness of the entire beam, which is divided into 8 finite elements with the piezo-patches placed at even finite element positions is assembled using the FEM technique and the assembled matrices (global matrices), $M$ and $K$ are obtained.

The equation of motion for the smart structure is finally given by [Manjunath and Bandyopadhyay, 2006a]

$$
M \ddot{q} + K q = f_{ext} + f_{ctrl} = f',
$$

(87)

where $M, K , q, f_{ext}, f_{ctrl}, f'$ are the global mass matrix, global stiffness matrix of the smart beam, the vector of axial displacements, transverse displacements and slopes, the external force applied to the beam, the controlling force from the actuator and the total force coefficient vector respectively.

The generalized coordinates are introduced into equation (87) using a transformation $q = T g$ in order to reduce it further such that the resultant equation represents the dynamics of the first 3 vibratory modes of the smart cantilever beam. $T$ is the modal matrix containing the eigenvectors representing the first 3 vibratory modes. The first 3 vibration modes $\omega_1, \omega_2$ and $\omega_3$, which are the most dominant modes compared to the other modes are considered in modelling the beam.

This method is used to derive the uncoupled equations governing the motion of the free vibrations of the system in terms of principal coordinates by introducing a linear transformation between the generalized coordinates $q$ and the principal coordinates $g$. Equation (87) now becomes

$$
M_T \ddot{g} + K_T g = f_{ext} + f_{ctrl1} + f_{ctrl2},
$$

(88)

where $f_{ctrl1}$ and $f_{ctrl2}$ are the vectors of the control force from the controller to the actuator. Multiplying equation (88) by $T^T$ on both sides and further simplifying, we get

$$
M^* \ddot{g} + K^* g = f^*_{ctrl1} + f^*_{ctrl2},
$$

(89)

where

$$
M^* = T^T M T, \ K^* = T^T K T, \ f^* = T^T f_{ext}, \ f^*_{ctrl1,i} = T^T f_{ctrl1}, \ i = 1 \ to \ 2.
$$

(90)

Here, $M^*, K^*, f_{ext}^*$ represent the generalized mass matrix, the generalized stiffness matrix, the generalized external force vector respectively. The generalized structural damping matrix $C^*$ is introduced into equation (89) as

$$
C^* = \alpha M^* + \beta K^*,
$$

(91)

where $\alpha$ and $\beta$ are the frictional damping constant and the structural damping constant used in $C^*$. The dynamic equation of the smart cantilever beam is obtained as
The generalized external force coefficient vector is
\[ M^* \dot{g} + C^* \ddot{g} + K^* g = f_{ext}^* + f_{ctrl}^* , \quad (92) \]
where \[ f_{ctrl}^* = f_{ctrl1}^* + f_{ctrl2}^* . \]

2.4 State Space Model of the Smart Structure

The state space model of the smart cantilever beam is obtained as follows.
\[ \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x} . \quad (93) \]

Now,
\[ \mathbf{g} = \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \quad (94) \]

and
\[ \ddot{\mathbf{g}} = \ddot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_4 \\ \ddot{x}_5 \\ \ddot{x}_6 \end{bmatrix} . \quad (95) \]

Thus,
\[ \dot{x}_1 = x_4 , \quad \dot{x}_2 = x_5 , \quad \dot{x}_3 = x_6 \]
and equation (92) now becomes
\[ M^* \ddot{x}_4 + C^* \dot{x}_4 + K^* x_4 = f_{ext}^* + f_{ctrl}^* , \quad (97) \]
which is further simplified as
\[ \begin{bmatrix} \ddot{x}_4 \\ \ddot{x}_5 \\ \ddot{x}_6 \end{bmatrix} = -M^{-1} K x_3 - M^{-1} C x_3 + f_{ext}^* + f_{ctrl}^* . \quad (98) \]

The generalized external force coefficient vector is
\[ f_{ext}^* = T^T f_{ext} = T^T f(r(t)) , \quad (99) \]
where \( r(t) \) is the external force input (disturbance) to the beam.

The generalized vector for the applied control force is
\[ f_{ctrl}^* = T^T f_{ctrl} = T^T h \cdot V_i^q(t) = T^T \mathbf{h} u_i(t) , \quad i=1 \text{ to } 2 , \quad (100) \]

where the voltages \( V_i^q(t) \) are the input voltages to the actuators 1 and 2 from the controllers respectively, and are nothing but the control inputs \( u_i(t) \) to the actuators, \( \mathbf{h} \) is a constant vector which depends on the actuator type, its characteristics and its position on the beam. Using equations (96), (99) and (100) in (98) and writing them in state space form, we get the state equation as
\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -M^{-1} K \\ -M^{-1} C \end{bmatrix} (6 \times 6) + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \]
\[ \begin{bmatrix} 0 \\ \mathbf{M}^{-1} \mathbf{T}^T \mathbf{h}_1 \\ \mathbf{M}^{-1} \mathbf{T}^T \mathbf{h}_2 \end{bmatrix} \mathbf{u}(t) + \begin{bmatrix} 0 \\ \mathbf{M}^{-1} \mathbf{T}^T \mathbf{f} \end{bmatrix} (6 \times 1) r(t) , \quad (101) \]
i.e.,
\[ \dot{\mathbf{x}} = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) + \mathbf{E} r(t) . \quad (102) \]

The sensor voltage is taken as the output and its equation (output equation) is modelled as
\[ V_i^s(t) = p_i^T \dot{q} = y_i(t) , \quad i=1 \text{ to } 2 , \quad (103) \]
where \( p_i^T \) is a constant vector which depends on the characteristics of the piezoelectric sensor. These include its type and its location in the embedding structure.

The sensor output is given by
\[ \begin{bmatrix} y_1^s(t) \\ y_2^s(t) \end{bmatrix} = \begin{bmatrix} 0 & p_1^T T \\ 0 & p_2^T T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} , \quad (104) \]

This equation can also be written as
\[ y(t) = C^T x(t) + D u(t) , \quad (105) \]
which is the output equation.
state vector, system output (sensor output). The numerical values of the state space matrices are given by

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.61 & 0.00 & -0.00 & -0.00 & 0.00 & -0.00 \\
0.00 & -2.13 & -0.00 & 0.00 & -0.00 & -0.00 \\
-0.00 & -0.00 & -5.39 & -0.00 & -0.00 & -0.00 & -0.0001
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0.0001 & -0.0001 \\
0.0000 & -0.0000 \\
0.0000 & -0.0031
\end{bmatrix}, \quad E = \begin{bmatrix}
0 \\
0 \\
0 \\
0.0037 \\
0.0000 \\
0.0015
\end{bmatrix}, \quad C^r = \begin{bmatrix}
0 & 0 & 0.0001 & -0.0 & -0.0000 \\
0 & 0 & 0 & 0.0000 & -0.00011
\end{bmatrix}, \quad D = \text{Null matrix},
\]

The control of this MIMO state space model is obtained using the multirate output feedback control technique based on POF, which is considered in the next sections. The characteristics of the smart MIMO cantilever beam are given in table 3.

<table>
<thead>
<tr>
<th>Position of PZT sensor / actuator</th>
<th>Eigen values</th>
<th>Natural frequency (Hz.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actuators 2,4</td>
<td>-0.3 ± j 246.43</td>
<td>39.2205</td>
</tr>
<tr>
<td>Sensors 6,8</td>
<td>-1.07 ± j 461.97</td>
<td>73.5245</td>
</tr>
<tr>
<td></td>
<td>-2.70 ± j 734.23</td>
<td>116.8571</td>
</tr>
</tbody>
</table>

3. CONTROL SYSTEM DESIGN

In the following section, we develop the control strategy for the multivariable representation of the MIMO smart structure model using the POF feedback control law [Werner and Furuta, 1995], [Werner, 1997] with 2 actuator inputs and 2 sensor outputs.

4.1 A brief review of the control technique

The problem of pole assignment by piecewise constant output feedback was studied by Chammas and Leondes [1979a, 1979b, 1979c], and Werner [1997] for LTI systems with infrequent observations. They showed that using a periodically time-varying piecewise constant output feedback gain [Levine and Athans, 1970], the poles of the discrete time control system could be arbitrarily assigned (within the natural restriction that they should be located symmetrically with respect to the real axis). Since the feedback gains are piecewise constants [Werner and Furuta, 1995], this method can easily be implemented and guarantees the closed loop stability. Such a control law can stabilize a larger class of systems. A brief review follows.

Consider a LTI CT system given by

\[
\dot{x} = \begin{bmatrix} A \\ B \end{bmatrix}x + \begin{bmatrix} Bu \\ 0 \end{bmatrix} , \quad y = \begin{bmatrix} Cx \\ 0 \end{bmatrix}, \quad (107)
\]

which is sampled with a sampling interval of \( \tau \) seconds given by the discrete Linear Time Invariant (LTI) system (called as the tau system) as

\[
x(k+1) = \begin{bmatrix} \Phi & \Gamma \end{bmatrix} x(k) + \begin{bmatrix} u(k) \end{bmatrix}, \quad (108)
\]

\[
y(k) = Cx(k),
\]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p \) and \( \Phi, \Gamma \) and \( C \) are constant matrices of appropriate dimensions. The following control law is applied to this system. The output \( y \) is measured at the time instants \( t = k\tau, k = 0, 1, 2, \ldots \)

We consider constant hold functions because they are more suitable for implementation. An output-sampling interval is divided into \( N \) sub-intervals of length \( \Delta = \tau / N \) and the hold function is assumed to be constant on these sub-intervals as shown in figure 2. Thus, the control law becomes:

\[
u(t) = \begin{bmatrix} K_1 y(k\tau) \\ \vdots \\ K_{i_N} y(k\tau) \end{bmatrix}, \quad (109)
\]

\[
(k\tau + l\Delta) \leq \tau \leq (k\tau + (l+1)\Delta), \quad K_{i_N} = K_1
\]

for \( l = 0, 1, 2, \ldots, (N-1) \).

Figure 2: Graphical illustration of POF control law
Note that a sequence of $N$ gain matrices \( \{ K_0, K_1, \ldots, K_{N-1} \} \), when substituted in (109), generates a time-varying piecewise constant output feedback gain $K(t)$ for $0 \leq t \leq \tau$.

Consider the following system, which is obtained by sampling the system in (107) at sampling interval of $\Delta = \tau / N$ seconds and denoted by \( (\Phi, \Gamma, C) \) called delta system:

\[
x(k+1) = \Phi x(k) + \Gamma u(k), \\
y(k) = C x(k),
\]  

(110)

Assume \( (\Phi, C) \) is observable and \( (\Phi, \Gamma) \) is controllable with controllability index $\nu$ such that $N \geq \nu$, then it is possible to choose a gain sequence $K_j$ such that the closed-loop system, sampled over $\tau$, takes the desired self-conjugate set of eigenvalues. Here, we define

\[
K = \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ \vdots \\ K_{N-1} \end{bmatrix},
\]  

(111)

then, a state space representation for the system sampled over $\tau$ is

\[
x(k\tau + \tau) = \Phi^N x(k\tau) + \Gamma u(k\tau), \\
y(k\tau) = C x(k\tau),
\]  

(113)

where

\[
\Gamma = \begin{bmatrix} \Phi^{N-1} \Gamma, \ldots, \Gamma \end{bmatrix}.
\]

Applying POF in equation (109), i.e., $K y(k\tau)$ is substituted for $u(k\tau)$, the closed-loop system becomes

\[
x(k\tau + \tau) = (\Phi^N + \Gamma KC) x(k\tau).
\]  

(114)

The problem has now taken the form of a static output feedback [Syrmos, 1997]. Equation (114) suggests that an output injection matrix $G$ can be found such that

\[
\rho (\Phi^N + GC) < 1,
\]  

(115)

where $\rho(\cdot)$ denotes the spectral radius. As \( (\Phi^N, C) \) pair is observable, one can choose an output injection gain $G$ to achieve any desired self-conjugate set of eigenvalues for the closed-loop matrix \( (\Phi^N + GC) \) and from $N \geq \nu$, it follows that one can find a POF gain which realizes the desired output injection gain $G$ by solving

\[
\Gamma K = G
\]  

(116)

for $K$.

The controller obtained from this equation will give the desired behaviour, but might require excessive control action. To reduce this effect, we relax the condition that $K$ must exactly satisfy the linear equation and include a constraint on it. Thus, we arrive at the following in the inequality equation

\[
\|K\| < \rho_1, \|\Gamma K - G\| < \rho_2.
\]  

(117)

Using the Schur’s complement, it is straightforward to bring these conditions in the form of linear matrix inequalities as

\[
\begin{bmatrix} -\rho_1^2 I & K \\ K^T & -I \end{bmatrix} < 0,
\]  

(118)

\[
\begin{bmatrix} -\rho_2^2 I & (\Gamma K - G) \\ (\Gamma K - G)^T & -I \end{bmatrix} < 0
\]  

(119)

In this form, the LMI toolbox of MATLAB can be used for the synthesis of $K$ [Yan et al., 1998], [Yang et al., 1993], [Geormel, 1994], [Gahnet, 1995]. The POF controller obtained by this method requires only constant gains and is hence easy to implement. Werner and Furuta [1995] and Werner [1997] proposed a performance index so that $\Gamma K = G$ need not be strictly imposed. This constraint is replaced by a penalty function, which makes it possible to enhance the closed-loop performance by allowing slight deviations from the original design and at the same time improving the behaviour. The performance index $J(k)$ is given by

\[
J(k) = \sum_{i=0}^{\infty} x_0^i u_i^T \left[ \begin{array}{cc} Q & 0 \\ 0 & R \end{array} \right] x_i + \sum_{k=1}^{\infty} (x_{kn} - x_{kn}^*)^T \bar{P} (x_{kn} - x_{kn}^*),
\]  

(120)

where $R \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times n}$, and $\bar{P} \in \mathbb{R}^{n \times n}$ are positive definite and symmetric weight matrices, $x_i$ and $u_i$ denote the states and the inputs of the delta system respectively and $x_{kn}^*$ denotes the state that would be reached at the instant $kN$, given $x_{(k-1)N}^*$, if $K$ is solved to satisfy (116) exactly, i.e., $x_{kn}^* = (\Phi^N + GC)x_{(k-1)N}$.
The first term represents the ‘averaged’ state and control energy whereas the second term penalizes the deviation of \( G \). A trade-off between the closed loop performance and the similarity to the chosen design is expressed by the above cost function.

4.2. Design of the MIMO POF Controller

An external force is applied to the free end of the beam and the OL response of the smart system is observed and shown in figures 3 and 5 respectively. A load matrix of unit sample is considered in the simulations. To dampen out the vibrations quickly, an external force is applied to the free end of the beam and the OL response of the smart system is observed and shown in figures 3 and 5, respectively.

4.2. Design of the MIMO POF Controller

The stabilizing output injection gain are obtained for the tau system such that the eigenvalues of \( \Phi^{\tau} + GC \) lie inside the unit circle and the response of the system has a good settling time. The output injection gain \( G \) is given by

\[
G = \begin{bmatrix}
-0.2095 & 0.0054 \\
0.0000 & -0.0000 \\
-0.0200 & 0.4930 \\
210.5816 & -5.4808 \\
0.0000 & 0.0000 \\
-15.1199 & -837.9257
\end{bmatrix}
\]  \quad (123)

The CL responses with the output injection gain are observed and shown in figures 3 and 5, respectively. Let \( (\Phi, \Gamma, C) \) be the discrete time matrices (delta system) for the beam shown in figure 1 and described in equation (106): their values are sampled at the rate \( 1/\Delta \) secs respectively and expressed as

\[
\Phi = \begin{bmatrix}
0.9982 & 0.0000 & 0.0000 \\
0.0000 & 0.9737 & -0.0000 \\
0.0000 & -0.0000 & 0.9814 \\
6.0251 & 0.0000 & 0.0000 \\
-0.0000 & 92.8365 & -0.0000 \\
0.0000 & 0.0000 & 71.4111 \\
-0.0001 & 0.0000 & 0.0000 \\
-0.0000 & -0.0004 & -0.0000 \\
-0.0000 & 0.0000 & -0.0001 \\
0.9982 & -0.0000 & -0.0000 \\
0.0000 & 0.9746 & 0.0000 \\
-0.0000 & -0.0000 & 0.9822
\end{bmatrix}
\]  \quad (124)

\[
\Gamma = 1e^{-6} \begin{bmatrix}
0.0000 & -0.0000 \\
-0.0000 & 0.0000 \\
0.0000 & -0.0001 \\
-0.0001 & 0.1092 \\
0.0000 & 0.0000 \\
-0.0010 & 0.4146
\end{bmatrix}
\]  \quad (125)

The number of sub-intervals, \( N \), is chosen to be 10. The periodic output feedback gain matrix \( K \) is obtained by solving \( \Gamma K = G \) using the LMI optimization method, which reduces the amplitude of the control signal \( u \) and is given by equation (126).

When the proposed controller is put in the loop, the closed loop impulse responses (sensor outputs \( y_1 \) and \( y_2 \)) with periodic output feedback gain \( K \) of the system are observed.

The variations of the control signals \( u_1 \) and \( u_2 \) with time for the MIMO model are also observed and they are graphically displayed in figures 4 and 6 respectively. The tip displacements are also observed. The comparisons of the quantitative
results of the OL and CL responses (with output injection gain, POF gain) with the magnitude of the control efforts, and their required settling times are shown in table 4.

\[ K = \begin{bmatrix} 0.0014 & 0.0529 \\ -1.2572 & -22.5487 \\ -0.0013 & -0.0552 \\ 1.0720 & 23.5198 \\ 0.0001 & 0.0422 \\ 0.9789 & -18.0520 \\ -0.0004 & -0.0171 \\ 0.3341 & 7.2781 \\ 0.0004 & -0.0135 \\ -1.0629 & 5.7890 \\ 0.0012 & 0.0411 \\ -1.2904 & -17.4790 \\ -0.0013 & -0.0580 \\ 0.9529 & 24.7388 \\ 0.0004 & 0.0594 \\ 0.8790 & -25.3775 \\ -0.0010 & -0.0442 \\ 0.6992 & 18.8384 \\ 0.0009 & 0.0163 \\ -1.2104 & -6.9271 \end{bmatrix} \]

Table 4: Quantitative comparative results of the POF simulations (terms inside the brackets indicate the settling values), only the positive values shown

<table>
<thead>
<tr>
<th></th>
<th>( y_1 ), FE 6 Sensor o/p</th>
<th>( y_2 ), FE 8 Sensor o/p</th>
</tr>
</thead>
<tbody>
<tr>
<td>OL</td>
<td>3.6 V 32 secs</td>
<td>1.8 V 11 secs</td>
</tr>
<tr>
<td>CL with ( G )</td>
<td>3.7 V 10 secs</td>
<td>1.8 V 9 secs</td>
</tr>
<tr>
<td>CL with ( K )</td>
<td>3.7 V 4 secs</td>
<td>1.9 V 8 secs</td>
</tr>
<tr>
<td>Control input ( u )</td>
<td>18 V for actuator at FE 2 and 39 V for actuator at FE 4</td>
<td></td>
</tr>
</tbody>
</table>

\[ \text{Control input } u = 18 \text{ V for actuator at FE 2 and } 39 \text{ V for actuator at FE 4} \]

5. SIMULATION RESULTS
The open loop impulse responses, the closed loop impulse response with the output injection and the periodic output feedback gain and the magnitudes of the control input required to damp out the vibrations are shown in figures 3 to 6 respectively.
6. CONCLUSIONS

A POF controller has been designed for the MIMO composite smart structure; when put in a feedback loop with the plant, the transient oscillations die out quickly in shorter times and steady state is reached quickly. It is also observed that modelling a smart structure by including the sensor / actuator mass and stiffness and by placing the sensor / actuator at two different positions introduces a considerable change in the structural vibration characteristics than placing the sensor / actuator pair at only one location [Manjunath and Bandyopadhyay, 2006a and 2006b].

The response takes shorter time to settle than the SISO case and the vibrations are damped out quickly. An overall better performance of the system is obtained as there will be multiple interactions of the input and the output which will cause the vibrations in the system to be damped out quickly. Responses were observed without control and were compared with the controlled ones to show the effect of control.

From the simulations, it was observed that without control the transient response was unsatisfactory and with control, the vibrations are suppressed. The shear and axial displacements, neglected in the Euler-Bernoulli beam theory, are considered in this research. The Timoshenko beam theory corrects the simplifying assumptions made in the Euler-Bernoulli beam theory, and the model obtained can be an exact one.

The designed POF controller requires constant gains and hence may be easy to implement in real time. A multi input multi output test provides better energy distribution and even better actuation forces then single input-output does. Unlike static output feedback, the POF control always guarantees the stability of the closed loop system. Surface mounted piezoelectric collocated sensors and actuators (piezo-patches bonded to the master structure at top and bottom of the single beam) are usually placed at the root of the structure (near the fixed end) as collocated pairs, one above and the other below the beam in order to achieve the most effective sensing and actuation.

The surface mounted sensors and actuators mounted on the beam are subjected to high longitudinal stresses that might damage the brittle piezo-electric material. Furthermore, surface mounted sensors / actuators are likely to be damaged by the contact of the piezo patches with the surrounding objects. Sometimes, the connecting leads attached to the piezo patches may come out while vibrating. The temperature, stray magnetic fields, noise signals, etc. may also have an effect on the performance of the piezo patches. The natural frequencies were found to be higher in the case of composite embedded beam than in the case of surface mounted ones. Embedded shear sensors / actuators, thus can be used to alleviate all the above mentioned problems.

APPENDIX

The stiffness matrix for the sandwich beam element is obtained using the equation (64) with coefficients specified as

\[ K_{11} = \frac{4 A_{11}}{L}, \quad K_{12} = K_{21} = \frac{-4 B_{11}}{L}, \quad K_{13} = K_{31} = 0, \]

\[ K_{14} = K_{41} = -\frac{4 A_{11}}{L}, \quad K_{15} = K_{51} = -\frac{4 B_{11}}{L}, \]

\[ K_{16} = K_{61} = 0, \quad K_{24} = K_{42} = -\frac{4 B_{11}}{L}, \]

\[ K_{34} = K_{43} = 0, \]

\[ K_{23} = K_{32} = 6 \frac{A}{5} - \frac{20 L^2 D_{21} \gamma L^2 + 60 A_{33}}{L}, \]

\[ K_{25} = K_{52} = -\frac{6 A}{5} - \frac{20 L^2 D_{21} \eta + 120 D_{21} \eta^2}{L}, \]

\[ K_{26} = K_{62} = \frac{1}{10} \frac{60 \gamma^2 A_{11} L^2 + 120 \gamma B_{11} \eta}{L}, \]

\[ K_{33} = \frac{360 A_{11} \eta L^2 + 180 \gamma L^2 B_{11} \eta + 45 \gamma^2 L^2 A_{11}}{15 L (12 \eta - L^2)}, \]

\[ A (2 L^4 D_{11} - 30 L^4 D_{21} \eta + 180 L^2 D_{21} \eta^2 - 15 L^2 \gamma B_{11} + 2160 A_{11} \eta^2 + 60 A_{33} L^2) - \]

\[ K_{33} = \frac{360 A_{11} \eta L^2 + 180 \gamma L^2 B_{11} \eta + 45 \gamma^2 L^2 A_{11}}{15 L (12 \eta - L^2)^2}, \]
\[ K_{35} = K_{33} = \frac{AL(D_1 L^4 - 10 B_i \gamma L^3 + 60 A_3 L +}{60 \gamma^2 A_1 L + 120 \gamma B_1 \eta}}{(12 \eta - L^2)^2} \]

\[ K_{36} = K_{33} = \frac{-A[L D_1 - 60 A_3 L^2 - 90 \gamma^2 L^2 A_1 -}{60 \gamma^2 D_1 \eta + 360 \gamma^2 D_1 \eta^2 +}{4320 A_5 \eta^2 - 720 A_6 \eta L^2]}{30 L (12 \eta - L^2)^2}, \]

\[ K_{44} = A \frac{A_{11}}{L}, \quad K_{45} = K_{54} = A \frac{B_{11}}{L}, \]

\[ K_{46} = K_{64} = 0, \]

\[ \{ D_1 L^4 + 10 A_3 L^2 + 10 \gamma^2 L^2 A_1 \}
\]

\[ K_{55} = 6 \frac{A}{5 L} - 20 \frac{L^2 D_1 \eta + 120 D_1 \eta^2}{(12 \eta - L^2)^2}, \]

\[ K_{56} = K_{65} = -\frac{1}{10} \frac{AL - D_1 L^4 - 10 B_i \gamma L^3 - 60 A_3 L}{(12 \eta - L^2)^2}, \]

\[ \{ A (2 L^2 D_1 + 15 L^2 \gamma B_1 - 30 L^2 D_1 \eta +}{60 A_3 L^4 + 45 \gamma^2 L^4 A_1 +}{-180 \gamma^2 L^2 B_1 \eta + 180 L^2 D_1 \eta^2 -}{360 A_5 \eta \gamma L^2 + 2160 A_6 \eta^2 \}
\]

\[ K_{66} = \frac{1}{15 L} \frac{(12 \eta - L^2)^2}{(12 \eta - L^2)^2}, \]

The mass matrix for the sandwich beam element is obtained using equation (60) with the coefficients as

\[ M_{11} = \frac{1}{3} L I_1, \]

\[ M_{12} = M_{21} = \frac{1}{2} \frac{\gamma L^2 I_1}{(12 \eta - L^2)^2}, \]

\[ M_{13} = M_{31} = -\frac{1}{4} \frac{\gamma L^2 I_1}{(12 \eta - L^2)^2}, \]

\[ M_{14} = M_{41} = \frac{1}{6} L I_1, \]

\[ M_{15} = M_{51} = \frac{1}{2} \frac{\gamma L^2 I_1}{(12 \eta - L^2)^2}, \]

\[ M_{16} = M_{61} = -\frac{1}{4} \frac{\gamma L^2 I_1}{(12 \eta - L^2)^2}, \]

\[ M_{22} = \frac{1}{35} \left[ 13 L^2 I_1 - 420 L \eta L^2 + 124 L^2 I_1 \eta + 42 \gamma^2 I_1 \eta L \right], \]

\[ M_{23} = M_{32} = \frac{1}{2} \frac{\gamma L^2 I_1}{(12 \eta - L^2)^2}, \]

\[ M_{24} = M_{42} = \frac{1}{2} \frac{\gamma L^2 I_1}{(12 \eta - L^2)^2}, \]

\[ M_{25} = M_{52} = \frac{3}{70} \left[ -84 I_3 \eta L^2 + 560 I_3 \eta^2 \right], \]

\[ M_{26} = M_{62} = \frac{420}{(12 \eta - L^2)^2}, \]

\[ M_{34} = M_{43} = -\frac{1}{4} \frac{\gamma L^2 I_1}{(12 \eta - L^2)^2}, \]

\[ M_{35} = M_{53} = \frac{1}{420} \frac{\gamma L^2 I_1}{(12 \eta - L^2)^2}, \]

\[ M_{44} = \frac{1}{3} L I_1, \quad M_{45} = M_{54} = -\frac{1}{2} \frac{\gamma L^2 I_1}{(12 \eta - L^2)^2}, \]

\[ M_{55} = \frac{1}{35} \left[ 1800 I_3 \eta \gamma^2 + 1260 I_2 \eta L^2 \right], \]

\[ M_{56} = M_{65} = \frac{1}{210} \frac{1800 I_3 \eta \gamma^2 + 12600 \eta^2 L^2}{(12 \eta - L^2)^2}, \]
\[
M_{ae} = \frac{[L (252 I, \eta^2 L^2 - 42 I, L; \eta + 63 \gamma^2 I, L^4
- 420 I, \eta L^2 + 2520 I, \eta^2 L - 210 I, \eta L^4
+ 10080 I, \eta^2 + 28 I, L^4 + 2 I, L^5)]}{210 (12 \eta - L^5)}.
\]

ACRONYMS

SISO Single Input Single Output
MIMO Multiple Input Multiple Output
FEM Finite Element Method
FE Finite Element
LMI Linear Matrix Inequalities
MR Magneto Rheological
ER Electro Rheological
PVDF Poly Vinylidene Fluoride
SMA Shape Memory Alloys
CF Clamped Free
CC Clamped Clamped
CT Continuous Time
DT Discrete Time
OL Open Loop
CL Closed Loop
HOBT Higher Order Beam Theory
RHS Right Hand Side
LTI Linear Time Invariant
FOS Fast Output Sampling
POF Periodic Output Feedback
AVC Active Vibration Control
EB Euler-Bernoulli
PZT Lead Zirconate Titanate
DOF Degree Of Freedom

NOMENCLATURE

\( A \) Area of the piezo patches
\( a_1, a_2, a_3, a_4 \) Polynomial coefficients for transverse displacement
\( A \) System matrix which represents dynamics of system (comprises of mass and stiffness of system)
\( A_{11}, A_{55} \) Extensional and shear stiffness coefficient
\( B_{11} \) Bending-extensional stiffness coefficient
\( B \) Input matrix
\( b_1, b_2, b_3 \) Polynomial coefficients for
\( c \) Width of the beam
\( c_1, c_2, c_3 \) Polynomial coefficients for axial displacement
\( C \) Output matrix
\( C^* \) Generalized damping matrix or the structural modal damping matrix
\( C_0 \) Fictitious matrix
\( D \) Transmission matrix
\( D_0 \) Fictitious matrix
\( D_{11} \) Bending stiffness coefficient
\( D \) Layer constitutive matrix
\( D_3 \) Electric displacement in the thickness direction
\( d_{15}, d_{31} \) Piezoelectric strain constants
\( E_{11} \) Actuator induced axial force
\( e_{15} \) Piezoelectric constant
\( E_f \) Electric potential applied to the actuator
\( E \) External load matrix, which couples the disturbance to the system
\( f'_{\text{ext}} \) Vector of externally applied nodal forces
\( f' \) Total force vector
\( f'_{\text{ctrl}} \) Control force vector
\( f^*_{\text{ext}}, f^*_{\text{ctrl}} \) Generalized external force coefficient and external control force coefficient vector
\( f^*_{\text{ctrl}1}, f^*_{\text{ctrl}2} \) Control force coefficient vectors to the actuators 1 and 2
\( F_{11} \) Actuator induced bending moment
\( F \) State feedback gain
\( G_{55} \) Actuator induced shear force
\( G_c \) Signal conditioning gain
\( G \) Modulus of rigidity
\( g \) Principal coordinates
\( h \) Height of the beam + the piezo-patches
\( h_1, h_2 \) Constant vectors of actuators 1, 2
\( I, I_x, I_z, I_3 \) Mass inertias
\( I \) Inertia matrix
\( i \) Variable (1, 2, 3, …)
\( i(t) \) Current induced by sensor surface
\( K, K^* \) Stiffness matrix (global stiffness matrix) and generalized stiffness matrix of the beam
\( k \) Variable (1, 2, 3, …)
\( K_c \) Gain of the controller
References:


AUTHOR BIOGRAPHIES

T. C. Manjunath, born in Bangalore, Karnataka, India on Feb. 6, 1967 received the B.E. Degree in Electrical Engineering from the University of Bangalore in 1989 in First Class and M.E. in Electrical Engineering with specialization in Automation, Control and Robotics from the University of Gujarat in 1995 in First Class with Distinction, respectively. He has got a teaching experience of 17 long years in various engineering colleges all over the country out of which, nearly 6 years in the research field. He is working as an Assistant Professor in the Department of Information Technology in Thakur College of Engineering and Technology, Kandivili, Mumbai, India since August 2006 and is also a Research Scholar in the department of Systems and Control Engineering, Indian Institute of Technology Bombay, India and progressing towards the completion of his Ph.D. in the field of modelling, simulation, control and implementation of smart flexible structures using dSPACE controller card and its applications. He has published 75 papers in the various national, international journals and conferences and published two textbooks on Robotics, one of which has gone upto the third edition and the other, which has gone upto the fourth edition along with the CD which contains around 200 C / C++ programs for various types of simulations on robotics. He is a student member of IEEE since 2002, SPIE student member and IOP student member since 2004, life member of ISSS, life member of Systems Society of India and a life member of the ISTE, India. His biography was published in 23rd edition of Marquis’ Who’s Who in the World in 2006 issue. He has also guided more than 2 dozen robotic projects. His current research interests are in the area of Robotics, Smart Structures, Control systems, Network theory, Mechatronics, Process Control and Instrumentation, CT and DT signals and systems, Signal processing, Periodic output feedback control, Fast output feedback control, Sliding mode control of SISO and multivariable systems and its applications.

Manjunath and Bandyopadhyay: Design of Multivariable POF Controller
B. Bandyopadhyay, born in Birbhum village, West Bengal, India, on 23rd August 1956 received his Bachelor’s degree in Electronics and Communication Engineering from the University of Calcutta, Calcutta, India, and Ph.D. in Electrical Engineering from the Indian Institute of Technology, Delhi, India in 1978 and 1986, respectively. In 1987, he joined the Interdisciplinary Programme in Systems and Control Engineering, Indian Institute of Technology Bombay, India, as a faculty member, where he is currently a Professor. He visited the Center for System Engineering and Applied Mechanics, Universite Catholique de Louvain, Louvain-la-Neuve, Belgium, in 1993. In 1996, he was with the Lehrstuhl fur Elektrische Steuerung und Regelung, Ruhr Universitat Bochum, Bochum, Germany, as an Alexander von Humboldt Fellow. He revisited the Control Engineering Laboratory of Ruhr University of Bochum during May-July 2000.

In May 2006, he was invited by the Systems Engineering Department, Okayama University, Japan for research collaboration. He has authored and co-authored 7 books and book chapters and more than 180 journal and conference papers. His research interests include the areas of large-scale systems, model reduction, nuclear reactor control, smart structure control, periodic output feedback control, fast output feedback control and sliding mode control. Prof. Bandyopadhyay served as Co-Chairman of the International Organization Committee and as Chairman of the Local Arrangements Committee for the IEEE International Conference in Industrial Technology, held in Goa, India, in Jan. 2000. His biography was published in Marquis’ Who’s Who in the World in 1997. Prof. B. Bandyopadhyay has been nominated as one of the General Chairman of IEEE ICIT conference to be held in Mumbai, India in December 2006 and sponsored by the IEEE Industrial Electronics Society.