Fast Algorithm for Simulation of Levy Stable Stochastic Self-Similar Processes

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Abstract—The paper proposes simulation method for modeling data flows needed in design and analysis of network information systems connected by various data channels. We propose a new fast algorithm of numerical simulation of stable self-similar Levy stochastic process. The self-similar property in paper means slow decrease of dispersion, long-range dependence and fluctuation character of a range of power of stochastic processes. The property of self-similarity is detailed and connected with property of stationarity of stochastic process in wide sense. The main result obtained in paper concerns with characteristic function of probability distribution for densities with infinite dispersion and expectation, called stable distributions. The numerical algorithm consists simple steps to obtain simulation of jump type stable stochastic self-similar processes increments.

Keywords—simulation algorithm; stable distributions; self-similar property; stochastic Levy process

I. INTRODUCTION

One of the major tasks in design and analysis of network information systems is creation of adequate mathematical models of network data flows, also named teletraffic models. It is known, that data network flows characterized, as rule, self-similarity property, which means slow decrease of dispersion, long-range dependence and fluctuation character of a range of power of such processes [1]. Existence of the listed properties at stochastic process means that its autocorrelation function matches autocorrelation functions of the aggregated processes precisely or asymptotically. Correlative properties of such process averaged on different time intervals, remain invariable. Levy stochastic processes are used in various fields, especially in financial modeling, and the most recent research have been carried out in [2]. Researches of the specified properties of teletraffic and new methods of its simulation for railway transport are considered in the monograph [3].

II. SELF-SIMILAR PROPERTY OF STOCHASTIC PROCESS

Let’s consider the stochastic processes, satisfying to self-similarity property: for everyone $a > 0$, depending on $a$, that

$$\text{Law}(X_{at}, t \geq 0) = \text{Law}(bX_{t}, t \geq 0).$$

Equation (1) means, that finite dimensional distributions of stochastic processes $X_{at}$ and $X_t$ are match. If in last equation value $b$ depends on $a$ such as $b = a^H$, then process $X$ is self-similar with Hurst exponent $H$. Thus $D = \frac{1}{H}$ is fractal dimension.

Let’s $X_t$ is stochastic process with self-similar property. Then let’s consider expectation $E[X_t(t > 0)]$. Since $\text{Law}(X_t) = \text{Law}(t^{H}X_t)$, then $E[X_t] = t^H E[X_t]$. Dispersion $DX_t$, by analogy follows $DX_t = t^{2H} DX_t$. If it is denoted $Y_t = E[X_t]$, and $Z_t = DX_t$, then in logarithmic scales we receive the linear dependences: $\ln Y_t = H \ln t + \ln E[X_t]$, $Z_t = 2H \ln t + \ln DX_t$, which explain determination of fractal dimensionality. Covariation function of self-similar stochastic process is:

$$\text{cov}(i, j) = E[X_t(X_{at} - EX_t)] = i^{2H} \text{cov}(1, i),$$

where $j = il$.

If it is denoted $\text{cov}(1, l) = \text{Cov}(l)$, then covariation function of self-similar stochastic process is $\text{cov}(i, j) = i^{2H} \text{Cov}(l)$, where $j = il$. Let us compare self-similarity property with stationarity property in a wide sense. We will remind [4] that stochastic process $X_t$ is stationary in wide sense (second order stationary) only case then:
1) there is an expectation $E[X_i]$ exists which doesn't depend on $i$;

2) covariance $\text{cov}(i, j) = E[(X_i - E[X_i])(X_j - E[X_j])]$ depends on value $k = |i - j|$, therefore $\text{Cov}(k) = \text{cov}(i, j) = \text{cov}(j, i)$, where $k = |j - i|$. Notice in that case dispersion $\sigma^2 = \text{Cov}(0)$ also doesn't depend on time, and then autocorrelation function is $R(k) = \frac{\text{Cov}(k)}{\sigma^2} = \frac{\text{Cov}(k)}{\text{Cov}(0)}$.

So, if self-similar process is stationary in a wide sense, as $H > 0$, $EX_t = DX_t = 0$ it follows $X_t = \text{const}$.

Let us compare the self-similar property with stochastic Levy processes [5]. Behavior of Levy processes is completely defined by univariate distributions: $F_i(x) = P(X_i \leq x)$ . It follows that $F_a(x) = P(X_{a_i} \leq x) = P(a^{H}X_i \leq x) = F_i(a^{-H}x)$, and particularly the family of distributions $F_i(x)$ is generated by a simple expression: $F_i(x) = F_i(t^{-H}x)$. If distributions $F_i(x)$ have probability densities, then family of distributions is generated by equation: $p_i(x) = t^{-H}p_i(t^{-H}x)$.

III. SIMULATION OF STABLE SELF-SIMILAR STOCHASTIC
LEVy PROCESSES

Let's set interval $[0,T]$, which is partitioned on $n$ elementary intervals $[t_{i-1}, t_i]_j$, $t_0 = 0$, $t_n = T$. Length of each interval is $h_n = \frac{T}{n}$. Let us consider the standard simulation interval $[0, 1]$. In accordance with interval $[0, T]$ partitioning we will get $[\tau_{i-1}, \tau_i]$, where $\tau_i = \frac{t_i}{T}$, and length of each part is $\eta = \frac{1}{n}$. Remember that stochastic process is self-similar therefore:

$\text{Law}(X_{\tau_i}, i = 1, \ldots, n) = \text{Law}(T^H X_{\tau_i}, i = 1, \ldots, n)$.

Further, having received selection on a standard interval, by means of the last ratio it is possible to receive selection on arbitrary interval $[0, T]$. From properties of Levy stochastic process it follows that random variables $(\Delta_i = X_{\tau_i} - X_{\tau_{i-1}})_{i=1}^n$ are independent and identically distributed, thus

$\text{Law}(\Delta_i) = \text{Law}(X_{\tau_i-\tau_{i-1}}) = \text{Law}(X_{\eta})$.

Denote $\phi(\theta) = E\exp(i \theta X_i)$ as characteristic function, then owing to homogeneity and independence of increments we will conclude that $\phi_{\alpha \Delta_i}(\theta) = \phi(\theta) \phi_{\alpha}(\theta)$. Solution of last equality is function $\phi(\theta) = \exp(\psi(\theta))$, where function $\psi(\theta)$ is cumulant. When $\tau = 1$, we will get, as result, function $\phi(\theta) = \exp(\psi(\theta))$ - characteristic function for $\text{Law}(X_1)$. From the self-similar property follows that law of probability distribution is stable with stable index $\alpha = \frac{1}{H}$. Thus, characteristic function is

$$\phi(\theta) = \begin{cases} \exp\left(i \mu \theta - \sigma^\alpha \theta^\alpha \left[1 - i \beta \text{sgn}(\theta) \lg \frac{\alpha \theta^\alpha}{2}\right]\right), & \text{if } \alpha \neq 1, \\
\exp\left(i \mu \theta - \sigma^\alpha \theta^\alpha \left[1 - i \beta \frac{2}{\pi} \text{sgn}(\theta)\right]\right), & \text{if } \alpha = 1,
\end{cases}$$

where $\alpha \in (0, 2)$, $\beta \in [-1, 1]$, $\sigma > 0$, $\mu \in R$.

The most interest case for simulation is symmetrical case when $\beta = 0$. In that case characteristic function is:

$$\phi(\theta) = \exp\left(-\sigma^\alpha |\theta|^\alpha\right).$$

If $\tilde{X}_i$ is self-similar stochastic Levy process with characteristic function $\tilde{\phi}(\theta) = \exp\left(-|\theta|^\alpha\right)$, then for process $X_i$ equation $X_i = \sigma \tilde{X}_i$ is correct.

Then we denote $\nu$ - standard normal distributed random variable, $\xi -$ is arbitrary independent random variable strictly higher zero, and characteristic function is

$$\phi(\theta) = \tilde{\phi}(\theta) = \exp\left(-|\theta|^\alpha\right).$$

We will compute the expectation $E\exp(i \theta \xi \nu)$ and use identity $E\exp(i \theta \xi \nu) = E(E(\exp(i \theta \xi \nu)/\xi))$. The conditional mathematical expectation is $E(\exp(i \theta \xi \nu)/\xi) = \exp\left(-\frac{(\theta \xi)^2}{2}\right)$. Let's $\lambda = \theta^2$, $\zeta = \frac{\xi^2}{2}$, and $E(\exp(i \theta \xi \nu)) = E(\exp(-\lambda \zeta))$. It follows that equation for unknown density of distribution of random variable $\zeta$ is:

$$E\exp(-\lambda \zeta) = \int_{0}^{\infty} \exp(-\lambda x) p(x) dx = \exp(-\lambda^\beta),$$

where $\beta = \frac{\alpha}{2}$. Thus, in (2) $\zeta$ is stable distributed random variable with stable index $\beta$, $0 < \beta < 1$. In [6] it is
determined that \( \zeta = \left( \frac{a(x, \beta)}{\delta} \right)^{1-\beta} \), where

\[
a(x, \beta) = \left( \frac{\sin \beta \chi}{\sin \chi} \right)^{1-\beta} \frac{\sin (1-\beta) \chi}{\sin \beta \chi}, \quad \chi \text{ and } \delta \text{ are independent random variables, } \chi \text{ is uniform distributed on interval } [0, 2\pi], \text{ } \delta \text{ is exponentially distributed with unit parameter.}
\]

Since for random variable \( \sqrt{2\zeta \nu} \) it is correct \( \text{Law}(\sqrt{2\zeta \nu}) = \text{Law}(\tilde{X}_1) \) we obtain the main result needed for simulation:

\[
\text{Law}(\Delta_i) = \text{Law}(\eta^H \sigma \sqrt{2\zeta \nu}). \tag{3}
\]

In (3) \( \Delta_i \) is independent identically distributed random variables and right side of (3) doesn’t depend on \( i \). The random variables \( \Delta_i \) are simulated based on three independent random variables \( \chi, \beta, \nu \), and random variables \( X_i = \sum_{k=1}^{\infty} \Delta_k \), where \( X_0 = 0 \).

IV. NUMERICAL SIMULATION ALGORITHM OF STABLE SELF-SIMILAR STOCHASTIC LÉVY PROCESS

1. Input parameters: \( \alpha, n \) (number of generated values), \( T \) (simulation interval), \( \sigma \) (dispersion of stochastic process).

2. Begin loop number 1 for \( i = 1, \ldots, n \).

3. Begin loop number 2 for \( k = 1, \ldots \).

4. Call procedure Compute \( \Delta_k \).

5. \( S := S + \Delta_k T \).

6. End of loop number 2 for \( k \).

7. End of loop 1 number for \( i \).

End of Main procedure

Procedure Compute \( \Delta_k \)

1. Input parameters: \( \alpha, n, \sigma \).

2. Compute \( \eta = \frac{1}{n} \).

3. Compute \( H = \frac{1}{\alpha} \).

4. Compute \( \beta = \frac{\alpha}{2} \).

5. Compute \( \Delta_k = 0 \).

6. Generate \( \chi = X_{U[0,1]} 2\pi \).

7. Generate \( \delta = X_{U[0,1]} \).

8. Generate \( \nu = X_{N(0,1)} \).

9. Compute \( \zeta = \left( \frac{a(x, \beta)}{\delta} \right)^{1-\beta} \),

where \( a(x, \beta) = \left( \frac{\sin \beta \chi}{\sin \chi} \right)^{1-\beta} \frac{\sin (1-\beta) \chi}{\sin \beta \chi} \).

10. Compute \( \Delta_k \) by (3).

End of procedure Compute \( \Delta_k \).

V. SIMULATION RESULTS

The figures 1 and 2 illustrate result of simulation stable self-similar stochastic Levy process with stable index \( \alpha = 0.75 \) and number of steps is 500 and 700 respectively.
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