Improvement Algorithm of Initial Iterative Value Selecting for Lambert Problem

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Abstract — For the Lambert problem of orbit transfer, this paper proposes an improvement solution based on the traditional flight-path angle iteration algorithm. First, a two-dimensional approximation function regarding to the transfer time, position vector and the flight-path angle is constructed by using the B spline interpolation function. Using the two-dimensional approximation function, the initial iteration value is redefined, which helps to improve the iteration efficiency. At last, the hypercritical test method and large number of experiments validate that compared to the traditional algorithm, the improvement flight-path angle iteration algorithm are more effective and faster in computation.

Keywords - Flight-Path Angel Iteration Algorithm, B Spline Interpolation, Approximation Function

I. INTRODUCTION

The orbit transfer can boil down to a classic astrodynamics problem—Lambert problem. For solving this problem, a lot of algorithms have been proposed, including Gauss’ universal variables method, Herrick method, Battin-Vaughan (B-V) method, flight-path angle iteration method, etc [1-6]. The original Gauss’ method is only effective for the range angle less than 90 deg. The universal variables method is sensitive to the initial values. And the (B-V) method requires complicate computation steps such as hypergeometric function solution, etc. In the orbit re-planning and interception for aircrafts, the common used method is the flight-path angle iteration method by Nelson-Zarchan. It uses the position vector and pose vector for computation, which makes the formula brief and intuitive [1-12].

This paper improvement the traditional flight-path angle iteration algorithm (may written as “traditional algorithm” hereafter) in the selection of the iterative initial value. Thus, the difference between the iterative value and the precise solution of the iterative equation set can be very small. The improvement flight-path angle iteration algorithm (may written as “improvement algorithm” hereafter) makes the whole solution process more scientific, improves the solution efficiency and saves a lot of computation time.

II. PROBLEM DESCRIPTION

The orbit transfer problem can be described as [1]: as shown in figure 1, assuming the initial position vector $r_1$, the final position vector $r_2$ and the flight time $t_F$ for the aircraft flying from $r_1$ to $r_2$ are known, the purpose is to find a transfer orbit for the aircraft flying from $r_1$ and $r_2$ with a flight time of $t_F$.

Under the geocentric coordinates system, the initial velocity of the aircraft should satisfy:

$$
|v_0| = \sqrt{\frac{f_m(1-\cos \phi)}{r_o \cos \gamma (\frac{r_o \cos \gamma}{r_o \cos \gamma - \cos (\phi + \gamma))}}}
$$

Where, $r_o$ is the initial geocentric distance and $r_0 = |r_0| = \sqrt{x_0^2 + y_0^2 + z_0^2}$; $r_f$ is the final geocentric distance and $r_f = |r_f| = \sqrt{x_f^2 + y_f^2 + z_f^2}$; $\gamma$ represents the initial flight-path angle; $\phi$ is the range angle between the initial position and the final position.

Flight time $t_F$ complies to the following formula:
In this formula, \( \lambda = \frac{r_y}{f \lambda} \). And for the eclipse trajectory, there is \( \lambda \ll 2 \).

As it can be seen from formula (1) and (2), the initial velocity \( dv \) and flight time \( F_t \) for arriving the final are only related to the initial flight-path angle \( \gamma \), if the initial position \( r_0 \) and the final position \( r_F \) are known. 

According to the Lambert guidance theory, by using the iteration algorithm, it is available to obtain the initial flight-path angle \( \gamma \) and initial velocity vector \( dv \) under the constraint of the flight time \( F_t \).

The selection of \( \gamma^{(0)} \) has a director effect on the number of iterations and convergence rate. In the traditional algorithm, the selection of \( \gamma^{(0)} \) is based on:

\[
\gamma^{(0)} = \left( \gamma_{\text{min}} + \gamma_{\text{max}} \right) / 2
\]

There are maximin constraints for the final flight-path angle \( \gamma \) of the aircraft. The range of \( \gamma \) is \( \left[ \gamma_{\text{min}} + \delta, \gamma_{\text{min}} - \delta \right] \) where \( \delta \) is a very small value greater than zero.

\[
\gamma_{\text{min}} = \tan^{-1} \left( \frac{\sin \phi - \sqrt{2r_y (1 - \cos \phi) / r_r}}{1 - \cos \phi} \right)
\]

\[
\gamma_{\text{max}} = \tan^{-1} \left( \frac{\sin \phi + \sqrt{2r_y (1 - \cos \phi) / r_r}}{1 - \cos \phi} \right)
\]

then,

\[
\gamma^{(1)} = \begin{cases} 
\left( \gamma_{\text{min}} + \gamma^{(0)} \right) / 2 & t^{(0)} > t_f \\
\left( \gamma_{\text{max}} + \gamma^{(0)} \right) / 2 & t^{(0)} \leq t_f
\end{cases}
\]

III. IMPROVEMENT OF THE ALGORITHM

The flight-path angle iteration algorithm is to solve the above equation set by gradual iterations. Significantly, an appropriate value is selected for the iteration initial value \( \gamma^{(0)} \). In this paper, the value of \( \gamma^{(0)} \) is redefined by the approximation function, which makes the initial value more approximate to the solution of the equation set.

A. Basic Idea of the Improved Algorithm

Let’s define the origin of the geocentric coordinates system is \( O \), the chord length from the initial point \( P_1 \) to the final point \( P_2 \) is \( c = \left| r_1 - r_2 \right| \) and half the perimeter of triangle \( \triangle OP_1 P_2 \) is defined as \( s = \left( r_1 + r_2 + c \right) / 2 \). \( \angle P_1 O P_2 \) is the geocentric angle of the transfer orbit, denoted by \( \phi \). Two none-dimensional parameters, \( \lambda \) and \( T \), are defined as:

\[
\lambda = \frac{1}{s} \sqrt{r_r c \cos \phi / 2}
\]

\[
T = \frac{8\mu}{s^3 t_f}
\]

According to the definitions of \( \lambda \), \( T \) and iterate variables \( x, y \) in the Battin-Vaughan algorithm (B-V algorithm), the following conclusion can be obtained—the precise solution of the equation set are only concerning to the none-dimensional parameter \( \lambda \) and \( T \) [8].

The semi-major axis \( a \) can be inferred:

\[
a = \frac{ms(1 + \lambda)^2}{8xy^2}
\]

The none-dimensional parameter \( \eta \) is defined as:

\[
\eta = \frac{s}{\mu} = \frac{8xy}{m(1 + \lambda)^2}
\]

Where,

\[
m = \frac{8\mu \cdot t_f}{s^3 (1 + \lambda)^n} = \frac{T^2}{(1 + \lambda)^n}
\]

As it can be seen from formula (9), \( \eta \) is merely concerning to \( \lambda \) and \( T \). Hence, for one set of input values \( \lambda, T \), one unique precise solution of \( \eta \) can be obtained. Namely, the variable \( \eta \) can be expressed as \( \eta = f(\lambda, T) \), a function of \( \lambda \), \( T \).

The transfer orbit from the original position \( r_0 \) and the final position \( r_F \) is a nominal eclipse orbit, which satisfies the eclipse orbit theorem:

\[
a = \frac{2 \mu r_0}{m^2 r_y}
\]

Based on formula (9) and (11), the expression of the initial velocity vector \( v_y \) can be obtained:

\[
\left| v_y \right| = \frac{2 \mu}{s} \sqrt{\frac{m^2}{r_y} - \frac{m^2}{s}}
\]
According to the above theory, the approximation function of $\eta$ can be constructed by using the sampling interpolation method, expressed as $\eta = g(\lambda, T)$. In the initialization process for $\gamma$, the following formula is used:

$$\eta^{(0)} = g(\lambda, T)$$  \hspace{1cm} (14)

Thus, the initial flight-path angle $\gamma$ is given by a superior initial value $\gamma^{(0)}$. Meanwhile, the value formula for $\gamma^{(1)}$ is optimized as :

$$\gamma^{(1)} = \gamma^{(0)} + \gamma^{(0)} * (t_T - t_T^{(0)})$$  \hspace{1cm} (15)

The adoption of the improved iterative initial value will greatly shorten the running time and improve the computation efficiency in the flight-angle iteration algorithm.

### B. Determination of the Approximation Function

Theoretically, for the given values of ($\lambda$, $T$), there is a corresponding value of $\eta$ as stated above. Thus, given a series of points ($\lambda_i$, $T_j$) ($i = 0,1,\ldots,N_\lambda$, $j = 0,1,\ldots,N_T$) and the corresponding $\eta_{i,j} = g(\lambda_i, T_j)$, a two-variable approximation function $\eta = Y(\lambda, T)$ can be constructed. If all the sampling points satisfy $Y(\lambda_i, T_j) = \eta_{i,j}$.

First, the variable $\lambda^*$ satisfies $0 < \lambda < 1$ and the value range of variable $T$ is $\eta$ large, spanning several orders of magnitudes. This is not benefit for the generation of the approximation function. Therefore, we have the following transformations for $\lambda$, $T$ and $\eta$ [8].

$$\lambda^* = \log_{10}(1+\lambda)$$  \hspace{1cm} (16)

$$T^* = \log_{10}(T)$$  \hspace{1cm} (17)

$$\eta^* = \log_{10}(\eta) = g(\lambda^*, T^*)$$  \hspace{1cm} (18)

![Figure 2. Relationship between $\gamma^*$ and $\lambda^*$](image2)

![Figure 3. Relationship between $\eta^*$ and $T^*$](image3)

The iterative variable $\lambda^*$ is nonlinear to the input variables ($\lambda^*$, $T^*$). The traditional equal interval sampling method cannot accurately reflect the correspondences between $\eta^*$ and ($\lambda^*$, $T^*$). Hence, nonuniform sampling is required for the independent variables ($\lambda^*$, $T^*$) in the section that the iterative variable $\eta$ changes violently. As shown in Fig. 2, "**" in the figure represents the relationship between the sampling point ($\lambda_i^*$, $T_i^*$) and the output $\eta_{i,j}^*$. The solid lines represent where the value of $T^*$ is the same. When $\lambda^* \in [0.15,0.25]$, the dependent variable $\eta^*$ changes violently. Therefore, $\lambda^*$ should be sampled with a higher density in the range of $[0.5,0.5]$. Shown as figure 3, "**" represents the sampling points and the solid lines represent where the value of $\lambda^*$ is the same. Therefore, $T^*$ should be sampled with a higher density in the range of $[0.3,0.5]$.

In the actual engineering calculation, the approximation function is always designed as a two-variable polynomial, expressed as $\eta = Y(\lambda, T)$. But $\eta = Y(\lambda, T)$ is highly nonlinear, the two-variable interpolation polynomial will have a relative higher order, which makes the calculation unstable. Therefore, this paper adopts the B spline interpolation function to solve the two-variable approximation problem [13-15].
The sampling points $P_j(\lambda_j, T_j)$ in inhomogeneous distribution are parameterized using the method of chord length cumulative.

\[
    u_i = u_{ij} = \frac{\sum_{k=1}^{i-1} |P_{k+1,i} - P_{k,i}|}{\sum_{k=1}^{i} |P_{k+1,i} - P_{k,i}|}, \quad u_{ij} = 0 \quad (19)
\]

\[
    2 \leq i \leq N_T, 1 \leq j \leq N_T
\]

\[
    v_j = v_{ij} = \frac{\sum_{k=1}^{j-1} |P_{i,k+1} - P_{i,k}|}{\sum_{k=1}^{j} |P_{i,k+1} - P_{i,k}|}, \quad v_{ij} = 0 \quad (20)
\]

\[
    1 \leq i \leq N_T, 2 \leq j \leq N_T
\]

The sapling points are transformed to the node parameters $(u_i, v_j)$, where $i = 0, 1, \cdots, n$, $j = 0, 1, \cdots, m$, $n = N_T$, $m = N_T$.

Rectangle domain $R = \{(u, v): a \leq u \leq b, c \leq v \leq d\}$.

\[
    \Pi_u : a = u_0 < u_1 < \cdots < u_n = b \quad \text{and} \quad \Pi_v : c = v_0 < v_1 < \cdots < v_m = d
\]

are the partition for the $u$-axis and $v$-axis of the parameter plane, respectively. The B spline function is denoted as:

\[
    f(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} u_{ij} \varphi_i(u) \psi_j(v) \quad (21)
\]

For the B spline function with node $u_i$, the space is denoted as $S(u: u_0, \cdots, u_n)$, the base of which is $\varphi_i(u)$. For the B spline function with node $v_j$, the space is denoted as $S(v: v_0, \cdots, v_m)$, the base of which is $\psi_j(v)$.

By approximation function and parameter transformation, we can obtain the relationship between the iterative variable $\eta$ and the input variables $(\lambda, T)$, expressed as $\eta = Y(\lambda, T)$.

Under the condition of low earth orbits, $(\lambda, T)$ are only concerning to the initial position $r_1$, the final position $r_2$ and the flight time $t_f$ and satisfies the constraints of $0 < \lambda < 1$ and $0.3 < T < 11$. Therefore, the interpolation approximation function $\eta = Y(\lambda, T)$ can be calculated by the preliminary treatment of the related data. In the iteration process for solving the Lambert problem in simulation, the following can be obtained with a single-step calculating using the approximation function.

\[
    \eta^{(0)} = Y(\lambda, T) \quad (22)
\]

Thus, the improvement iterative initial value $\gamma^{(0)}$ can be calculated out quickly.

IV. SIMULATION EXAMPLES AND ANALYSIS

A. Simulation Examples

Neglecting the perturbations of the environment and space of earth, simulations of interception guidance are conducted by using the traditional flight-path angel iteration and the improvement algorithm, respectively.

Assume the initial position the aircraft $r_1 = 7500km$, final position $r_2 = 6371km$, range angle $\phi = 50^\circ$ and flight time $t_f = 1100s$. And the parameters of the transfer orbit from $r_1$ to $r_2$ can be calculated out: the semi-major axis $a = 5280km$ and semi-parameter $p = 4287km$.

Figure 4. The comparison of two algorithms in number of iterations

In the traditional algorithm, $\gamma = 0.1176rad$, the computation time is 0.9ms and the number of iterations is 5. Under the same simulation conditions, using the improvement algorithm in this paper, the computation time can be shortened to 0.11ms with the same calculation precision by 3 iterations.

The iterative initial value calculated by the improvement algorithm is $x^{(0)} = 0.1184rad$. The difference with the precise solution of the iteration equation set is only $0.0008rad$. That is to say, the calculated initial value is very close to the solution of the equation set $\gamma$, shown in Fig. 4, which saves plenty of time for the iterative calculation.

B. Statistical Analysis

Assuming the initial position of the interceptor is $r_1 = 7500km$, $p$ defined as $p = r_2 / r_1$, samples 20 points ($N_p = 20$) in $[0.8, 1.25]$ at the initial position $r_1 = 7500km$, $p = r_2 / r_1$. The transfer angle $\phi$ samples 20 points ($N_{\phi} = 20$) in $[40^\circ, 60^\circ]$. Meanwhile the flight time $t_f$ samples 20 points ($N_{t_f} = 20$) in $[1000s, 2500s]$. The sample points
construct the simulation examples, the total number of which is Table I.

| Table I. Statistics about the number of iterations of the simulation examples |
|-----------------------------------|-----------------------------------|
| Iteration times | Traditional flight-path angle iteration algorithm | Improvement flight-path angle iteration algorithm |
| Number of examples | Cumulative rate % | Number of examples | Cumulative rate % |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 54 | 0.68 |
| 3 | 3 | 0.04 | 815 | 10.86 |
| 4 | 116 | 1.49 | 4621 | 68.63 |
| 5 | 1633 | 21.9 | 2510 | 100 |
| 6 | 2990 | 59.28 | 7 | |
| 7 | 3250 | 99.9 | 2 | |
| 8 | 8 | 100 | 3 | |

In our experiment, the actual total number of the simulation examples is 8000. The traditional flight-path angle algorithm and the improvement algorithm are adopted respectively for all the simulation examples. The number of iterations and the running time of each simulation example are recorded carefully and put into statistical analysis, shown as Table 1 and Table 2. By the comparison of the traditional algorithm and the improvement algorithm, the following conclusions can be obtained: the average iterative times of the traditional algorithm is 6.174 and the improvement algorithm is 4.198, which only account for 2/3 of the traditional algorithm. And, under the same conditions, the average running time of the traditional algorithm is 0.107ms and maximum running time is 3.6ms while the average running time of the improvement algorithm is 0.081ms and maximum running time is 0.384ms.

| Table II. Statistics about the running time of the simulation examples |
|-----------------------------------|-----------------------------------|
| Running time | Traditional flight-path angle iteration algorithm | Improvement flight-path angle iteration algorithm |
| Number of examples | Cumulative rate % | Number of examples | Cumulative rate % |
| 0 0 0 0 | 0 | 8000 | 8000 |
| 80 | 0 | 3803 | 47.54 | 6434 | 80.43 |
| 60 | 0 | 3970 | 97.16 | 1554 | 99.85 |
| 40 | 0 | 202 | 99.69 | 10 | 99.98 |
| 20 | 0 | 12 | 99.84 | 2 | 100 |
| 10 | 0 | 6 | 99.91 | 6 | |
| 0.6 0 | 0.04 | 5 | 99.98 | 5 | |
| 0.6 0 | 0 | 2 | 100 | 2 | |

The above comparison show that, the improvement flight-path angle iteration algorithm are obviously superior to the traditional flight-path angle iteration algorithm both in iteration times and running time.

Then hypothesis testing (simulation examples \( n = 8000 \) ) is performed based on the average values regarding to the iteration times and the running time of the data populations of the two algorithms. Because the data sources of the two algorithms come from the same sample points and the iteration processes are similar, the two data populations can be regarded as matched population samples. For this matched samples, the following hypothesis testing is performed:

Because the data sources of the two methods come from the same sample points and the iteration processes are similar, the two data sample can be regarded as matched samples. The population T expressed as the difference about calculating time of these matched samples which are between traditional method and improvement method. \( X_i, i = 1, \ldots, n \) (simulation examples \( n = 8000 \) ) is the sampling examples in population T. Thus, there are the hypothesis testing of computation time as follows.

\[
H_0 : \mu_c > 0 \quad H_1 : \mu_c < 0 \quad \alpha = 0.05
\]

where \( t_t = \frac{\bar{X}_c - \mu_c}{S_{X_c}} \),

\[
S_{X_c} = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X}_c)^2}{n-1}} = 5.7536 \times 10^{-4}.
\]

\( \bar{X}_c \) is the expectation of the averages of the population where \( X_i \) is in \( \mu_c \). \( \mu_c \) is the critical expectation.

Because of \( t = 39.1276 > -1.6449 \) , the test statistic \( t_t \) accepts hypothesis \( H_0 \), which means the improvement method is superior to the traditional method in running time.

Doing the same thing in iteration times. The population C expressed as the difference about number of iteration of these matched samples. So the hypothesis testing as

\[
H_0 : \mu_t \geq 0 \quad H_1 : \mu_t < 0 \quad \alpha = 0.05
\]

where \( t_t = \frac{\bar{X}_c - \mu_t}{S_{X_t}} \),

\[
S_{X_t} = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X}_t)^2}{n-1}} = 0.74251.
\]

\( \bar{X}_t \) is the expectation of the averages of the population where \( X_i \).
is in.

Because of $t_c = 237.978 > -1.6449$, the test statistic $t_c$ accepts hypothesis $H_0$, which means the improvement method is superior to the traditional method in iteration times.

The hypothesis tests show that, the improvement algorithm is greatly superior to the traditional algorithm both in number of iterations and running time. Though the above simulation experiment, it can be proved that, the improvement algorithm has the same computation precise with the traditional algorithm while the improvement algorithm has better initialization value formula, higher computation efficiency.

IV. CONCLUSION

For the problem of orbit transfer, the flight-path angle iteration algorithm by Nelson-Zarchan is the most widely used in the calculations about near-earth aircrafts because it is intuitive and simple.

This paper proposes an improvement flight-path angle iteration algorithm. First, the approximation function $\eta = Y(\lambda, T)$ regarding to the input variables $(\lambda, T)$ and iteration variable $n = \frac{\eta}{a}$ is constructed. On this basis, a new method for initializing $\gamma$. At last plenty of simulations validate that, the improvement algorithm is easy to use and more effective in computation compared to the traditional algorithm.

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