A Study on Nonlinear Forced Vibration of an Axial Moving Viscoelasticity Beam using the Multi-Scale Approach

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Abstract — The multi-scale method is used to analyse the steady-state response amplitude of a nonlinear forced vibration of axial moving viscoelasticity beam, and the Galerkin truncation method is applied to solve its numerical differential equation by discretization. The influences on the steady-state response amplitude due to various factors such as: i) varied viscosity damping coefficient, ii) external excitation, iii) nonlinear coefficients and iv) moving velocity are analysed. The results agree with those in the literature, i.e. the external motivation and viscous damping coefficient have great effect on the dynamics behaviour of the vibration system. The results are then derived for different order Galerkin truncations and compared, and the instability boundary conditions are determined. The results show that the unstable condition is mainly induced by the external excitation and viscosity damping, which agrees well with the results in the literature. Our study provides reliable theoretical basis for the operation of flexible devices with security and stability.

Keywords: Axial moving beam, multi-scale method, Galerkin truncation, steady-state response.

I. INTRODUCTION

Axially moving viscoelastic beam, as a simplified physical model is widely used in a number of systems, such as hoist, power transmission belts, hand saws, aerial ropeway etc. Under the action of external excitations, these systems may produce unwanted vibrations and resonances, causing considerable damage in the normal operation. The vibration is related to the system's flexible structure, which can be approximatively regarded as the beam with viscoelastic properties. Therefore, studies on the nonlinear forced vibration of axial moving viscoelastic beam has caught many scholars' eyes  [1-5].

The governing equation for the longitudinal vibrations of viscoelastic moving body is a typical continuous system, containing mixed partial derivative terms of both time and space [6]. After the research on transverse vibration of axially moving elastic beam by Pasin [7], a number of scholars devoted themselves to focusing on vibration problems. Among the studies reviewed, most vibration problems are simplified based on the Kirchhoff theory. Oz [8] discussed the plane vibration of an elastomer in viscous medium and obtained the relationship between free vibration frequency and damping. Liu [9] established the dynamic equation of vibration with rod centerline as any plane curve and discussed the plane vibration problems of elastic rod with circular cross section in viscous medium. Due to the limitation in statics category, the Kirchhoff theory is not sufficient to calculate properly when solving the continuum movement problems. Then, a series of methods such as the differential quadrature, finite difference were developed and tested, which make the nonlinear responses of axially moving beam under parametric excitations be acquainted by us. But there is still disagreement between axial tension of moving beam and speed pulsation[10]. Galerkin truncation method was recently used to study the stability and dynamics of axially moving beam with simple support. Its basic principle is to overlap the finite polynomial function and get a series of algebraic equations suitable for certain boundary conditions. Ghayesh [11] discussed the branch and chaotic characteristics of axial moving body under the longitudinal forced vibration and plane nonlinear vibration which coupled the radial and transverse direction. Ding et. al [12] studied the chaotic dynamic behavior of viscoelastic beam with parametric excitation which is caused by external harmonic excitation and speed pulsation, and observed the doubling bifurcation phenomenon of axially moving beam on disturbance velocity amplitude by Poincare mapping.

The multi-scale approach (MSA) is applied to determine the amplitude-frequency characteristics and steady-state response of nonlinear forced vibration when approximately solving the axially moving continuum issue in order to keep high accuracy and low computational cost. The vibration is a multi-scale coupled phenomenon and the main features of the objective world complexity, namely, coherent structure and random fluctuations appear simultaneously or alternately and coupled with each other, which determines that it cannot be explained from a single point of view[13]. Currently, many scholars investigated the vibration problems of moving body from the angle of multi-scale. Chen [14] analyzed the stable boundary of axial shift viscoelastic beam under clamped conditions through the average approach and got the stable boundaries of clamped beam by using the multi-scale approach. Yao [15] studied the nonlinear dynamic behavior of axial moving beam with parametric excitation under multi-pulse excitation by combining the multiple time scales method and Galerkin truncation method. Pakdemirli [16] applied the multi-scale approach to the partial differential equations and studied the stability of accelerated beam when the stiffness is small.
In this paper, the multi-scale approach is applied to solve governing equations and the different truncation Galerkin method is used to study the nonlinear forced vibration of axial moving viscoelastic beam. The influences of viscous damping, external excitation, nonlinear coefficient and moving velocity on steady-state response amplitude are studied and abundant amplitude response result are obtained, meanwhile, the boundary conditions when instability is occurred are also analyzed.

II. GOVERNING EQUATION

Kelvin model has been widely used in the axial moving beam. After Mockensturm et al. [17] deemed that the physical time should be taken into consideration to express the energy dissipation during steady movement, Chen et al [18] applied the physical time derivative in viscoelastic properties of axially moving beam. There was a simply clamped axially moving beam with uniform density $\rho$ and cross-sectional area $A$. And the length between supports $L$, initial tension $P_0$, axial velocity $v$, excitation amplitude $F$. The model was shown in Fig. 1, in which cross-sectional area moment of inertial $I$, elastic ratio is $E$ and viscosity coefficient is $\eta$. The control equation can be show as Equation (1) when considering the lateral vibration in the plane only, where the displacement $X$ and force $F$ is the function of axial coordinate $x$ and time $t$.

$$\rho \frac{\partial^2 U}{\partial t^2} + 2\eta \frac{\partial U}{\partial t} + \frac{dV}{dx} \frac{\partial U}{\partial x} + g \frac{\partial^2 U}{\partial x^2} = F$$

(1)

Boundary conditions of simply supported beam at both ends:

$$U(0, t) = 0, \quad \frac{\partial U}{\partial x} \bigg|_{x=0} = 0$$

$$U(L, t) = 0, \quad \frac{\partial U}{\partial x} \bigg|_{x=L} = 0$$

(2)

Bring in the dimensionless variables and parameters:

$$\frac{u}{U} = \frac{x}{L}, \quad \frac{t}{T} = \frac{t}{\sqrt{\rho A L^2}}, \quad \gamma = \frac{\sqrt{\rho A L}}{P_0}, \quad \nu = \frac{EL}{P_0}, \quad k = \frac{I\eta}{E\sqrt{\rho A L}}$$

(3)

where $\epsilon$ is an small dimensionless parameter. The dimensionless form of Equation (1) and (2) can be show as Equation (4) and (5).

$$\frac{\partial^2 u}{\partial t^2} + 2\eta \frac{\partial u}{\partial t} + \gamma \frac{\partial^2 u}{\partial x^2} + (\gamma^2 - 1) \frac{\partial^2 u}{\partial x^4} + \nu \frac{\partial^2 u}{\partial x^2} + k \frac{\partial^2 u}{\partial x^4} = f$$

(4)

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x} \bigg|_{x=0} = 0, \quad u(L, t) = 0, \quad \frac{\partial u}{\partial x} \bigg|_{x=L} = 0$$

(5)

III. MULTI-SCALE APPROACH

The multi-scale approach can be used in solving the nonlinear partial differential equation problem approximately. Under the simply supported boundary conditions, the multi-scale approach is employed to solve the partial differential equations (4), with the time scale as $T\epsilon^2$. The equation can be written by:

$$u(x, t; \epsilon) = u_0(x, T_0, T_1) + \epsilon u_1(x, T_0, T_1) + \cdots$$

(6)

where $u_0$ and $u_1$ represent the fast scale and slow scale displacement function respectively. $T_0 = t$ and $T_1 = \epsilon t$ are separately the fast time scale and slow time scale. Inserting Equation (4) into Equation (6) and separating the different orders, we get:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial T_0} + \epsilon \frac{\partial u}{\partial T_1} + \cdots$$

(7)

When Equation (7) is substituted in Equation (4), one obtains Equation (8) and (9) by $u_0$ and $u_1$.

$$\frac{\partial u}{\partial T_0} + 2\eta \frac{\partial u}{\partial T_0} + \frac{dV}{dx} \frac{\partial u}{\partial x} + (\gamma^2 - 1) \frac{\partial^2 u}{\partial x^4} + \nu \frac{\partial^2 u}{\partial x^2} + k \frac{\partial^2 u}{\partial x^4} = f$$

(8)

$$\frac{\partial u}{\partial T_0} + 2\eta \frac{\partial u}{\partial T_0} + \frac{dV}{dx} \frac{\partial u}{\partial x} + (\gamma^2 - 1) \frac{\partial^2 u}{\partial x^4} + \nu \frac{\partial^2 u}{\partial x^2} + k \frac{\partial^2 u}{\partial x^4} = 2\eta \frac{\partial^2 u}{\partial x^2}$$

(9)

The solution of Equation (8) is given by Wickert and Mote [19]:

$$u_0 = \sum_{i=0} u_i(x) \phi_i(T_0) e^{i\omega_i T_1} + \bar{u}_i(x) \overline{\phi_i(T)} e^{-i\omega_i T_1}$$

(10)

where $\omega_i$ and $\phi_i$ represent the $i$-th natural frequency and the corresponding complex modal function of linear homogeneous system respectively. The $\phi_i(x)$ that meet the condition of both-clamped beam can be expressed as
represent the amplitude and phase angle of the nonlinear free vibration separately. And the real and imaginary parts of \( A_i \) can be shown as:

\[
\begin{aligned}
an_i &= f[\text{Im}(\gamma)]\sin(\psi) - \text{Re}(\gamma)\cos(\psi)] + \nu\text{Re}(\mu)\eta_i = \phi(a_i, \gamma_i) \\
\gamma_i &= f[\text{Im}(\gamma)\cos(\psi) + \text{Re}(\gamma)\sin(\psi)] + \omega_1 + \text{Im}(\kappa)\kappa_i = \psi(a_i, \gamma_i)
\end{aligned}
\]

where \( \sigma \) is the harmonic parameter. In order to analyze the amplitude response near the resonance point, the relationship between disturbance frequency \( \omega \) and natural frequency \( \omega_i \) can be written as Equation (20). And the relationship between amplitude response and harmonic parameter is obtained by eliminating the phase, as shown Equation (21).

\[
\omega = \frac{\omega_i}{3 + \epsilon \sigma}
\]

(19)

In order to easily calculate, the Galerkin truncation method is adopted to convert the differential equation of continuum system into the ordinary differential equation with finite dimension and the dynamic characteristics of axially accelerating viscoelastic beam is analyzed numerically, where the displacement variables can be represented as,

\[
u(x, t) = \sum_{i=1}^{N} q_i(t) \cdot \phi_i(x)
\]

(22)

\[
F(x) = \sum_{i=1}^{N} f_i \phi_i(x)
\]

(23)

The discretization equations will constrain quickly with the increase of \( i \), where \( q_i \) stands for the generalized displacement function and \( f_i \) means the \( i \)-th forced vibration amplitude which is a constant value. Inserting Equation (22) and (23) into Equation (13), multiply by the corresponding characteristic function and convert into the discrete function \([0, 1]\), which can express as matrix:

\[
I\ddot{q} + C\dot{q} + Kq + Mq = N
\]

(24)

where \( I, C, K, M \) and \( N \) represent the mass matrix, the damping matrix, the stiffness matrix, the external stiffness
matrix and the external excitation matrix separately.

\[
q = [q_1(t) q_1(t) \cdots q_1(t)], \phi = [\phi(t) \phi(t) \cdots \phi(t)]
\]  

(25)

\[
I = \int_0^1 \phi \phi^T dx
\]  

(26)

\[
C = 2\gamma \int_0^1 \phi \phi^{(4)}T dx + k \int_0^1 \phi^{(4)} dx
\]  

(27)

\[
K = (\gamma^2 - 1) \int_0^1 \phi \phi^T dx + \nu^2 \int_0^1 \phi^{(4)} dx + \gamma k \int_0^1 \phi^{(4)} dx
\]  

(28)

\[
M = \frac{3}{2} \nu^2 \int_0^1 \phi \phi^{(4)}T dx + 2\gamma k \int_0^1 \phi \phi^{(4)}T dx + \gamma k \int_0^1 \phi^{(4)} dx
\]  

(29)

IV. RESULT ANALYSIS

A. Model Validation

Fig. 2 shows the change of lateral vibration displacement in the center of axially moving viscoelastic beam with time under the system parameters set as viscous damping coefficient \(k=0.00015\), external excitation \(f=0.0008\), moving velocity \(\gamma=4.5\), and nonlinear coefficient \(v=100\). It is observed that the vibration displacement is stable when loading periodically. However, the periodical changing of displacement is not the key to measure the vibration characteristics. In this paper, the investigation focus is the variation of amplitude, namely, the relative value between the maximum and minimum displacement variation.

In order to verify the feasibility of the multi-scale model, Fig. 3 gives the quantitative comparisons among the multi-scale calculation, differential quadrature method (DQM)\(^{20}\) and finite difference method (FDM)\(^{21}\) with parameters as \(k=0.00015\), \(f=0.0001\), \(\gamma=4\), and \(v=100\). It is clear that by the above three methods, the curves that represent the change of the steady-state response amplitude with harmonic parameters show great agreement. Thus, it is feasible to employ the multi-scale method to simulate the vibration of viscoelastic beam. Under the same parameters, the values of the steady state response amplitude got by multi-scale method are between that of DQM and FDM, which results from the different dispositions at the nonlinear parameter terms. And the multi-scale method can obtain relatively accurate result by contrast.

B. Forced Vibration Analysis

The steady state response of vibration amplitude is affected by macro performance, such as viscous damping coefficient \(k\), external excitation \(f\), moving velocity \(\gamma\) and nonlinear coefficient \(v\). Fig. 4(a) shows the relationship between steady state response amplitude and viscous damping coefficient with \(f=0.0001\), \(\gamma=4\), and \(v=100\). It can be seen that the steady-state response amplitude decreases with the increase of damping coefficient and this variation is obvious near the natural frequency \(\omega_i(\sigma=0)\). In other words, the viscosity play a role to absorb vibration, and that’s why flexible structure is used in many transportation systems with dynamic load. For example, the wire rope in hoisting system. The inconspicuous relationship between variation of steady-state response amplitude and nonlinear coefficient is depicted in Fig. 4(b) with \(k=0.0001\), \(f=0.0001\), and \(\gamma=4\). Further, the law that the steady-state response decreases with the increase of moving velocity is obtained in Fig. 4(c) with \(k=0.0001\), \(f=0.0001\), and \(\gamma=4\). It is found that higher velocity can improve the dynamic balance of system and aggravate the bifurcation phenomenon, which agrees well with the results in literature\(^{10-22}\).

The relationship between the steady-state response amplitude and external excitation is studied with \(k=0.0001\), \(\gamma=4\), and \(v=100\). As shown in Fig. 4(d), the steady state response amplitude will increase when the external
incentive becomes intense, which agrees well with the conclusion in test [23].

The influence of external incentives on the steady-state response amplitude is obvious when the values are far away from the natural frequency $\omega_i$. However, the external motivation is inevitable in the practical engineering and how to regulate the system parameters becomes a key in system designing. To conclude that lower external motivation, higher viscous damping coefficient and moving velocity can effectively restrain vibration, which is useful for the moving of axial viscoelasticity beam.

C. Vibration Stability

Fig.5 (a) and (b) show the comparisons of the real part of steady-state response amplitude when the order of Galerkin truncation is 1, 2, 4 under the parameters as $k=0.00005, f=0.0012, \gamma=3, \nu=50$ and $k=0.0003, f=0.0008, \gamma=4, \nu=200$ respectively. It is observed that the 1 order Galerkin truncation can produce reliable primary resonance, but 2 and 4 order Galerkin truncations show close results. That is to say, the difference among natural frequencies of axial moving beam can be obtained by Galerkin truncation method, which is reflected by steady-state response of different amplitudes in the critical region [20]. Then, the instability boundary of main resonance is analyzed and Fig.6 (a) and (b) show the results of steady-state response amplitude and the instability boundaries under the condition of $k=0.00005, f=0.00015, \gamma=5, \nu=100$ and $k=0.0003, f=0.0005, \gamma=5, \nu=200$ respectively, where the solid line represents the stable region, the dotted line represents the instability boundary, and the imaginary line represents the instability region. By comparison, although the moving velocity and nonlinear coefficient is higher, the response amplitude can always keep stable if the viscous damping coefficient increase and the external excitation become
weak, which is consistent with the conclusion in reference [22]. The results indicate that over small viscous damping coefficient and overlarge external excitation will cause instability.

![Graph](image1)

Fig. 5 Comparison of different orders Galerkin truncation

This is because the viscous has the power to absorb vibration to keep balance, and the external incentives enhance the occurrence of chaos and bifurcation [9]. It is known that instability will influence the running stability of the flexible systems and the unwanted phenomena like skidding may occur with the overlarge moving displacement of nonlinear moving beam due to the aggravation of instability. Therefore, the choice of parameters in practical applications should be reasonable to avoid the occurrence of instability.

![Graph](image2)

Fig. 6 Steady-state response and the instability boundaries.

V. CONCLUSIONS

In this paper, the nonlinear forced vibration of the axially moving viscoelastic beams are researched by using the multi-scale approach, together with Galerkin truncation method to numerically solve its differential equation, and obtain the reliable conclusions. Firstly, the accuracy of the model was verified through the quantitative comparison with the differential quadrature method and finite difference method. Then, the influences of viscous damping coefficient, external motivation, nonlinear coefficient and moving velocity on the steady-state response amplitude were analyzed. The steady-state responses amplitude is inversely proportional to the value of viscous damping coefficient and moving velocity, and proportional to external excitation. All above results are consistent with the references. At last, by comparing the different order Galerkin truncation, the instability boundary conditions are analyzed, and found that the too small viscosity or too large external motivations cause instability. The research that multi-scale approach and Galerkin truncation method are applied to explore the nonlinear forced vibration of the axially moving viscoelastic beam is very useful. The
conclusions have great value in theory and also can guide us to control instability effectively through optimizing parameters.

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