An Application of Nonlinear Functional Analysis to the Simulation of Nonlinear Control System

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Abstract — Nonlinear systems are modelled in the mathematical domain using a set of simultaneous equations and are approximated by linearization. In this paper we utilize functional analysis to improve the linearization of nonlinear control equations. Improved linearization increases the controllability of the system. Banach spaces are utilized in achieving the desired controllability. The proposed technique is designed and implemented for a single input single output system.

Keywords — Nonlinear control systems, functional analysis, Linearization, Controllability, Banach spaces.

I. INTRODUCTION

Most of the scientific, engineering and physical phenomena in practice today are based on strong mathematical modeling and simulation before the actual testing and implementation takes place. The proposed research paper deals with a nonlinear class of systems with an efficient application of functional analysis theory for achieving controllability of the nonlinear relationship. Control systems are classified as linear and nonlinear based on the relationship between the inputs given to the system and the response obtained from the system. In other words, linear control systems go by the principle of superposition and are modelled and governed by a set of linear differential equations. Simple examples of linear control systems could be a purely resistive network powered by a constant DC source of voltage which generates a linear voltage [3] and current in response to the linear input. The conditions of additivity and homogeneity are satisfied by linear control systems. Fundamental models exhibiting nonlinear characteristics could be the magnetization curve which is initially linear but exhibits non-linearity as it reaches saturation. A similar kind of characteristic could be observed in DC motor curve under a no load condition. Essential parameters of a control system are its controllability, stability and observability. Considering a system with the following control equations defined as:

\[ x' = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) \]

Where \( x(t) \in \mathbb{R}^n \) is defined as the state vector and \( u(t) \) is defined as the input to the system with \( y(t) \) being the response of the system.

Based on the definition of the linear system illustrated in (1) and (2), observability is defined as a condition when equilibrium of the system achieves the condition:

\[ x^e(t) = x(t, x^e, 0) \text{ for all } t \geq 0 \]

And the controllability of the system is defined as the state in which for every initial condition denoted by \( x(0) \) and input vector, there exists a time instant \( t \) such that:

\[ x(t, x(0), u) = x^e \]

Where \( x^e \) denotes the final state of the system.

Another important parameter with respect to a general control system is the stability of the system where a controllable system is said to be stable when the governing equation takes the form:

\[ u(t) = Mx(t) + l(t) \]

where \( M \) is defined as the feedback matrix.

A simple illustration of general control system is shown in figure 1. The above illustration contains both the linear and nonlinear part. The linear part is the output \( C(s) \) while the nonlinear part is the feedback from the output which is used to modify the response coefficients of the system to minimize the error to achieve convergence as early as possible.

There are a number of factors which affect the performance of the control system especially in the nonlinear model. Linearization has a direct consequence on the performance of the system. Linearization could be achieved through several approaches. Gain scheduling is one of the common approaches where the system under consideration is linearized around some operating conditions. Around this stable point, the parameters are taken to be constant and gain control mechanism is used.
to bring about linearization. Other methods found in the literature include linear matrix inequalities where the given nonlinear optimization problem is viewed as a convex optimization problem. These type of techniques utilize the $L_2$ norm parameter to bring about the desired linearization. A drawback of this approach observed in the literature is that it requires knowledge about the structure of the system under consideration. Transform based approaches have also been utilized in the past where linear fractional transforms [LFT] [12] [18] [19] have been used which go by a non-convex optimization approach. In terms of stability, inversion techniques like linear dynamic inversion techniques have been utilized in the literature to bring about the desired stability to the system. Lyapunov based stability approaches have also been used in the past to bring about desired stability to the system by appropriately choosing the gain of the system to be sufficiently large based on the bounds. Stability is achieved by going in from BIBO principles which states that for every bounded input a bounded output generates stability. Stability is also achieved by using linear parameter varying models which come under the category of robust dynamic inversion techniques.

As mentioned in previous sections, three important parameters control the efficiency of the system as studied from the literature [16]. They are Controllability, Observability and Stability of the model. Since, the proposed work is focused on nonlinear control; controllability and stability have been regressively researched and studied in the literature. Controllability as the name indicates denotes the degree of control or steering that the central unit of the control system has on its behavior [4] [9] [14] [17]. The essential difference in controllability in linear and nonlinear systems is that in nonlinear systems, the system controls the reach of final state from the initial state where the input and output vary over a wide range. Literature indicates the works of Balachander et al, Jetto et al and Yong Li et al whose works have attempted to transform the controllability problem [8] to a set of fixed point problems in the subspace over which the system is modelled in the vector domain. The works of Wang JinRong and Zhou Yong have investigated a fractional [19] as well as semi linear differential control equations by utilizing the popular Banach spaces for providing optimality to the problem objective. Some hybrid system have also utilized Bohnenblust-Karlin’s [6] fixed point theorem along with Banach spaces to further optimize the stability as well as controllability of the system. The works of Jin rong et al [3] have presented findings of two necessary and sufficient conditions for improving controllability over fractional and semi linear systems. Optimal set of mild class solutions have also been provided in the literature which could be applied to both linear and nonlinear systems.

Literature also reveals utilization of implicit theorems, impulse functions and Sobolev equation models for optimization of semi linear fractional systems. The obtained solutions are also separable in the sub spaces. Findings in the literature also indicate a condensed mapping algorithm [7] [13] being applied to achieve controllability and stability to the nonlinear system under study. Other works in Banach spaces indicate algorithms which achieve control in neutral differential control equations using a hybrid Dhage fixed point theorems. A vast investigation of controllability [5] of first order, second order and integro differential equations have been found in the literature including the works of Yeoul Park et al and Benchorra et al on the second order differential equation nonlinear model. Researches [4] [8] have also investigated the effect of infinite delay on the objective function. Further works indicate the analysis of stochastic models in addition to infinite delay in the system. The works of Sakthivel et al provide a new set of necessary and sufficient conditions to approximate the controllability of fractional semi linear differential equation models. Hilbert space [17] has been utilized to achieve the solution for optimality.

This paper is organized as follows. Section II describes the proposed methodology and analysis followed by conclusion in section III.

II. PROPOSED WORK

The objective of the proposed research work is to obtain an optimal solution for control of nonlinear control systems in which the function spaces are defined as:

$$P := O^ω(0, T; R)$$

And

$$Q := M^1,ω(0, T; R)$$

Based on the above set of nonlinear unconstrained bounded variables defined in (6) and (7), the solution is achieved and defined as:

$$C_{min} = X(u, v) := \int_0^t l(u(t), z(t))dt + \Phi(y(t))$$

A. Functional Analysis Theory

Functional analysis is an important mathematical tool for approximating most of the everyday scientific and engineering application for linear solutions such that the control of the nonlinear differential equations governing the system under study could be easily achieved. Vectors, Subspaces and subsets form integral part of the functional analysis theory. Functional analysis theory derive their inspiration from Fourier analysis and transform in defining the properties of continuous signals and systems. Functional analysis theory finds massive utilization in systems involving differential and integral control.
B. Banach Space

A general mapping of a set $S: A \subseteq Y \rightarrow A$ could be defined in Banach space such that $S(x) = x$ where $S$ is the contraction constant in the Banach space to maintain the controllability of the system. If $S$ is a closed subset of Banach space $Y$ and $A \rightarrow A$ defined as a contraction in the subspace then a unique fixed point function could be defined in the Banach space as:

$$|S^\alpha(x) - \bar{x}| \leq \frac{\phi}{1-\phi} |S(x) - x| \quad (9)$$

Here we define for a continuous system that:

$$|\bar{x}(y + \nu) - \bar{x}(y)| = |S(\bar{x}(y + \nu), y + \nu) - S(\bar{x}(y), y + \nu) + S(\bar{x}(y), y + \nu) - S(\bar{x}(y)), y) \leq |\Phi| \bar{x}(y + \nu) - \bar{x}(y)| + S(\bar{x}(y)), y + \nu) - S(\bar{x}(y)), y) \quad (10)$$

Simplifying the above inequality it could be concluded that:

$$|\bar{x}(y + \nu) - \bar{x}(y)| \leq \frac{\phi}{1-\phi} |S(\bar{x}(y)), y + \nu) - S(\bar{x}(y)), y)\vert \quad (11)$$

Extending the above equation for a continuous solution we can prove that:

$$\bar{x}(y + \nu) - \bar{x}(y) = \bar{x}'(y) = r(t) \quad (12)$$

If $A, B, C$ are spaces defined in Banach domain and $X, Y$ are subsets of $P$ and $Q$ respectively then:

$$F \in A_\alpha(X \times Y, C), r \geq 1 \quad (13)$$

Utilizing the shift operation, $A \rightarrow A - A(p, q)$ where $A(p, q)$ is taken to be zero and hence the solution converges to the equation:

$$P(x, y) = y - cS(\xi(y), y)^{-1} \quad (14)$$

The solutions oscillate around $p$ and $q$. The final solution for convergence is achieved by repeated iteration of the expression:

$$x = y - cS(\xi(y), y)^{-1}S(x, y) \quad (15)$$

Using the implicit function theorem the solution of controllability could be achieved as:

$$\bar{y} = cS(\bar{y}, \lambda), y \in F^{n+1} = \{y \in G^{n+1}(N, N), p(x) = 0\} \quad (16)$$

The zeros in the above control equation for analysing the stability of the system is given as:

$$Z: G^{n+1}X(X(-N, N) \rightarrow G^{n+1}(-N, N) (x, \lambda, \epsilon) \rightarrow \bar{x} - \epsilon F(x, \lambda) \quad (17)$$

The experimentation has been done in MATLAB for a SISO system modelling a Wiener filter to remove the noise from the input signal. The processor is an Intel 3 2.5 GHz processor with a 4GB RAM capacity. Figure 2 illustrates the nonlinear system modelled in SIMULINK environment with known input parameters to each of the blocks. The output signal is viewed across the scope. In case of second order differential equations of non-linear systems, cosine functions act as essential tools in linearization of the model.

Figure 2. Nonlinear model developed in Simulink

In the proposed work, the infinite dimensional Banach spaces depict the abundance of spectral properties exhibiting a L2 stability and asymptotic stability. But results depict that more the nature of the curve to be exponential, more the stability of the system. Sufficient conditions have been derived to guarantee the exponential stabilizability of the zero solution of a perturbed linear
time-varying control system in infinite-dimensional Banach spaces. These results become explicit to control systems with compact and self-adjoint coefficients on a separable Hilbert space.

Figure 3 illustrates the non-linear control points plotted in the three dimensional subspace in MATLAB. Figure 4 depicts the linearized output from a non-linear model utilizing a wiener filter for removal of noise. It could be seen that the relationship between controllability and stabilizability depicted in finite-dimensional spaces exhibits an exponential stability. The rising exponential curve could also be seen from figure 5. It could be clearly understood that null controllable system exhibit exponential stability in the infinite dimensional space. The linearized output in figure 4 clearly illustrates the close following of the linearized output with that of the original noisy signal. This close tracking helps in quick convergence of error in the feedback which is non-linear as mentioned in previous sections. This quick convergence approximates the non-linear properties down to linear approximated output.

Figure 3 Location of nonlinear control points in 3D space.

Figure 4 Linearized output of a SISO model.
Figure 5 illustrates the convergence of error where the point of stability is achieved. A near to convergence is achieved at about 20 – 25s which is a satisfactory convergence rate with respect to non-linear systems.

Figure 5 Error convergence of the non linear model.

III. CONCLUSION

This paper presents a novel nonlinear control strategy for a class of uncertain single-input and single-output discrete-time nonlinear systems with unstable zero-dynamics utilizing the applications of functional analysis theory for the nonlinear control. Banach spaces have been utilized to optimize the solution for the nonlinear system. The proposed method combines Banach spaces with cardinality with multiple models, and it has been shown that the nonlinear system can ensure the boundedness of the input and output signals and the nonlinear control equations can improve the dynamic performance of the closed loop system. This method provided optimality by relying on the assumption of commonly-used uniform boundedness on the un modeled dynamics which helps in increasing the applicability of the nonlinear functional analysis theory. Moreover, regressive subspace analysis and implicit theorem have been used to estimate and compensate the effect caused by the un modeled dynamics which appear in the form of convergence rate error quantity arising from the feedback of the output node. Utilization nonlinear functional analysis theory in Banach spaces ensures a perfect one to one mapping which is justified from the proof of the solutions used in optimizing the SISO system.

REFERENCES