Research on the Numerical Analysis of Nonlinear Neutral Functional Differential Equations

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Abstract — In this paper, the author studies the numerical analysis of nonlinear neutral functional differential equations. Functional differential equations (FDEs) arise widely in physics, biology, engineering, medical science, economics and so on. It is meaningful to study the theory and application of numerical methods for FDEs. The result shows gets B-stability, B-consistency and B-convergence results of numerical methods which are more general and deeper than the existing related traditional method.

Keywords - Numerical Analysis; Nonlinear Neutral Equation; Functional Differential Equations.

I. INTRODUCTION

Functional differential equations (FDEs) arise widely in physics, biology, engineering, medical science, economics and so on. It is meaningful to study the theory and application of numerical methods for FDEs. In the last several decades, many important results on the theory of computational methods for FDEs have been reported by a large number of researchers. Especially, the general stability theory and B-theory of numerical methods including Runge-Kutta methods and general linear methods for nonlinear stiff Volterra functional equations (VFDEs) in Banach spaces has been established in recent years, which provides a unified theoretical foundation for the stability of the theoretical solution and B-stability and B-convergence of numerical methods for nonlinear stiff problems in delay differential equations (DDEs), integral-differential equations (IDEs), delay-integral-differential equations (DIDEs) and VFDEs of other types that appear in practice. However, that work isn’t adapted for nonlinear neutral FDEs, FDEs, a class of hybrid systems, which consist of functional differential equation and functional equation and are much more extended than neutral functional differential equation. Discussing the theoretical solution and numerical methods are more complex than other functional differential equations.

At present, several authors only have investigated the asymptotic stability of numerical methods for linear FDEs. In view of this, Shang’s paper [1] investigates the stability of the theoretical solution and B-stability and B-convergence of Runge-Kutta methods for a class of nonlinear FDEs with variable delay. The research obtains the stability, general contractility and asymptotic results for a class of nonlinear FDEs with variable delay. Wang [2] applies Runge-Kutta methods to nonlinear FDEs with variable delay; he gets B-stability, B-consistency and B-convergence results of numerical methods which are more general and deeper than the existing related results in literature.

In probability theory, this type equation is connected with stochastic process via Kolmogorov forward equations. The foundation of the solution was represented by the central result in the modern theory of stochastic processes, Feynman-Kac formula.

However, the simulation for the statistical representation of Feynman-Kac formula needs many painstaking efforts, especially the annoying frequent change of the initial condition. Then many mathematicians and physicists pay attention to fundamental solution, which plays an important role in the research of Ref. [3], the author discussed the existence, uniqueness and the boundary of the fundamental solution. Gronwall’s inequality, Hamack’s inequality, the maximum principle are the commonly used and effective technology. We will follow the assumptions by Friedman to guarantee these basic properties of fundamental solution. In this dissertation, we dedicate to the duality of Ito diffusion and the factorization and simulation of the fundamental solution for parabolic type equation of Cauchy problem. By this simulation, to get the solution we only need to calculate the integral which is much easier than other methods. Furthermore, we must point out that the fundamental solution can be treated as adjusted transitional function by the relationship with the transitional probability density function of diffusion.

In general the terminal condition of the BSDE may be a general function of Brownian paths. In this situation the solution (F, Z) is regarded as a generalized "path-dependent" solution of the above BSDE and it has long been discussed that this BSDE can also be viewed as a path-dependent PDE (PPDE). This problem was raised by the Ref. [4].

II. THE BASIC WORK AND MODEL FOR NUMERICAL ANALYSIS

Neutral functional differential equations (NFDEs) can be found in many scientific and technological fields such as biology, physics, control theory, engineering and so on. In the last four decades, basic theory of NFDEs and numerical methods for NFDEs has been widely discussed by many authors because of its importance. On the other hand, due to the difficulty of the research, so far stability analysis of the theoretical and numerical solutions are still limited to linear problems and several classes of special nonlinear problems in literature.
Agarwal [5] introduces the test problem classes $L^\lambda \neq (\alpha, \beta, \gamma, \tau, 1, \tau 2)$ and $D^\lambda \neq (\alpha, \beta, \gamma, \tau, 1, \tau 2)$ with respect to the initial value problems of nonlinear NFDEs in Banach spaces. A series of stability, contractibility, asymptotic stability and exponential asymptotic stability results of the theoretical solutions to nonlinear NFDEs in Banach spaces are obtained. These results are the basis of numerical stability analysis in Grace’s paper [6]. It should be pointed out that the aforementioned results can be regarded as an extension of the stability theory of nonlinear stiff Volterra functional differential equations (VFDEs) in Banach spaces established by Hale [7]. Other related results we have seen in literature are all limited to the stability analysis of the theoretical solutions to neutral delay differential equations (NDDEs) in a finite dimensional spaces.

Han [8] studies the stability and convergence properties of numerical methods for nonlinear NDDEs in Banach spaces. Sufficient conditions for the stability and asymptotic stability of linear $\theta$-methods for a class of NDDEs with many delays are obtained, and a series of stability results are obtained for a class of linear multistep methods with variable coefficient and for several classes of explicit and diagonal implicit Runge-Kutta methods when applied to nonlinear NDDEs with variable delay. Moreover, convergence results of a class of linear multistep methods with variable coefficient are also obtained.

Up to now only a few papers in literature have researched the numerical stability of several classes of special nonlinear NDDEs in finite dimensional spaces. Using a one-sided Lipschitz condition together with some classical Lipschitz conditions, Han gives the error estimation of one-leg methods and waveform relaxation methods (WRM) for nonlinear NDDEs with variable delay in a finite-dimensional space. We consider three different approaches to approximating neutral term, prove that a one-leg method with linear interpolation is $E$ (or $EB$)-convergent of order $p$ if and only if it is $A$-stable and consistent of order $p$ in the classical sense for ODEs, where $p = 1, 2$. We also give the convergence results on waveform relaxation methods. Several numerical tests are given that confirm the theoretical results mentioned above.

A series of stability and asymptotic stability criteria of $G(c, p)$-algebraically stable one-leg methods and $(k, l)$-algebraically stable Runge-Kutta methods for a class of nonlinear neutral delay integro-differential equations (NDIDEs) are obtained. Using a one-sided Lipschitz condition, we also obtain the convergence results of $G$-stable one-leg methods and algebraically stable Runge-Kutta methods for the class of nonlinear NDIDEs. The researches have done a series of numerical experiments which confirm the theoretical results mentioned above. As far as we know, there are a few papers dealt with the linear numerical stability for NDIDEs.

Some sufficient conditions for the dissipativity of the solutions to neutral differential equations with piecewise constant delay and bounded variable delay are obtained. For neutral differential equations with piecewise constant delay, we prove that under one of the following two conditions $1. A$-

III. RESULTS AND DISCUSSION

The basic equation is shown in the following equation (1):

$$[x(t) + p(t)x(\tau(t))]^{(\alpha)} + q(t)x^{\alpha}(\sigma(t)) = 0$$

(1)

Since the differential equations have important applications in the natural sciences, technology and population dynamics, there is a permanent interest in obtaining sufficient conditions for the oscillation or nonoscillation of the solutions of various types of even-order/odd-order differential equations; see references in this article, and their references.

If $x$ is a positive solution of (1), then the corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ satisfies:

$$z(t) > 0, z^{(n-1)}(t) > 0$$

(2)

Assume that $\alpha \geq 1, c, d \in R$. If $c \geq 0, d \geq 0$, then

$$c^\alpha + d^\alpha \geq \frac{1}{2^{\alpha-1}}(c + d)^\alpha$$

(3)

By the definition of convex function, we have

$$f(c + d) \leq \frac{f(c) + f(d)}{2}$$

(4)

That is,

$$\frac{1}{2^{\alpha-1}}(c + d)^\alpha \leq c^\alpha + d^\alpha$$

(5)

So we get the equation (3). This completes the proof. Next, we establish our main results. For the sake of convenience, let

$$Q(t) = \min \{q(t), q(\tau(t))\}$$

(6)

Assume that

$$\int_{t_0}^{\infty} t^{\alpha-1} Q(t) dt = \infty$$

(7)

Further, assume that the first-order neutral differential inequality

$$\left(\frac{1}{b} \left(\frac{p^n}{b} y(\tau(t))\right) + \frac{Q(t)}{2^{\alpha-1}}\right)\left(\frac{\lambda}{(n-1)!}\right)^{\alpha-1} y^{\alpha}(\sigma(t)) \leq 0$$

(8)

The most common form and its expression is as follows:
In view of this, the exponential model is employed to estimate the variogram function, and its expression is shown in Formula (10).

\[
c(L) = \begin{cases} 
0 & L = 0 \\
C_i \left(1 - \exp\left(-\frac{3L}{a}\right)\right) & L \geq 0 
\end{cases}
\]  (10)

Where C1 denotes the semi-variance of the known data points in the neighborhood when interval \( L > 3a \), then:

\[
\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c}
C_{11} \\
C_{12} \\
C_{13} \\
C_{14}
\end{array}\right] =
\left[\begin{array}{c}
C_{21} \\
C_{22} \\
C_{23} \\
C_{24}
\end{array}\right] =
\left[\begin{array}{c}
C_{31} \\
C_{32} \\
C_{33} \\
C_{34}
\end{array}\right]
\left[\begin{array}{c}
C_{41} \\
C_{42} \\
C_{43} \\
C_{44}
\end{array}\right]
\]  (11)

The expression of the Green function is expressed in Formula (12) and (13).

\[
\left\{ \begin{array}{ll}
\frac{1}{a} e^{r^2} \sigma^2 \
0
\end{array} \right. \quad r \neq 0 \quad r = 0
\]  (12)

\[
\left[\frac{1}{a} e^{r^2} \sigma^2 + (pr + 1) \right] \
0
\]  (13)

In the comparison principle in (7) we do not assume that the deviating arguments are either delay or advanced type, and hence this result is applicable to all types of equations. Further, the comparison principle established in (7) reduces oscillation of equation (1) to find conditions for the first-order neutral differential inequality (5) has no positive solution. Therefore, applying the conditions for equation (5) to have no positive solution, one can immediately get oscillation criteria for equation (1).

For such kind of material, the general form of equation (10) is expressed as following equation (13-14):

\[
G_{ik}(k, \omega) = \frac{1}{\rho_0 \omega^2} \left\{ \frac{\beta^2}{k^2 - \beta^2} \right\} + \sum_{i=1}^{n} \omega_i \left\{ \left(1 - \frac{e^{r^2} \sigma^2}{r^2} \right) \right\}
\]  (12)

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\]  (13)

\[
\gamma_{ik}(k, \omega) = \frac{1}{\rho_0 \omega^2} \left\{ \frac{e^{r^2} \sigma^2}{\eta_1^2} \right\} \frac{\beta^2}{k^2 - \beta^2} \frac{m_i}{C_{44}}
\]  (14)

In which,

\[
\alpha^2 = \frac{\rho_0 \omega^2}{C_{11}^0} \quad \alpha^2 = \frac{\rho_0 \omega^2}{C_{44}^0} \quad \beta^2 = \frac{\rho_0 \omega^2}{C_{11}^2}
\]  (15)

The first one is the function for random dynamic systems in the form of

\[
\frac{1}{e^{-\varepsilon^2 - \sigma^2}} g(r, t) = \delta(t) \delta^2(r)
\]  (16)

Another function is defined as:

\[
\frac{1}{e^{-\varepsilon^2 - \sigma^2}} h(r, t) = \delta(t) \delta^2(r)
\]  (17)

The function component defined for Fourier transform can be obtained in the following equation (19)-(21).

\[
G_{ik}(r, \omega) = \frac{i}{4\rho_0 \omega^2} \left\{ \frac{\beta^2}{k^2 - \beta^2} \right\} + \sum_{i=1}^{n} \omega_i \left\{ \left(1 - \frac{e^{r^2} \sigma^2}{r^2} \right) \right\}
\]  (12)

\[
G_{ik}(r, \omega) = \frac{i}{4\rho_0 \omega^2} \left\{ \frac{\beta^2}{k^2 - \beta^2} \right\} + \sum_{i=1}^{n} \omega_i \left\{ \left(1 - \frac{e^{r^2} \sigma^2}{r^2} \right) \right\}
\]  (13)

\[
\gamma_{ik}(r, \omega) = \frac{i}{4\rho_0 \omega^2} \left\{ \frac{e^{r^2} \sigma^2}{\eta_1^2} \right\} \frac{\beta^2}{k^2 - \beta^2} \frac{m_i}{C_{44}}
\]  (14)

In which,

\[
\Theta(t - \frac{r}{c_i})
\]  (22)

Meanwhile, it also represents output for equation (5)


\[ \bar{h}(r, t) = \frac{\Theta(t - \frac{r}{c})}{2\pi} \]

(23)

\[ \begin{array}{l}
\left\{ t \ln \left( \frac{ct + \sqrt{c^2t^2 - r^2}}{r} \right) - \sqrt{r^2 - \frac{r^2}{c^2}} \right\}
\end{array} \]

It can be represented as:

\[ c_1 = \sqrt{\frac{C_{11}^0}{\rho_0}}, \quad c_2 = \sqrt{\frac{C_{66}^0}{\rho_0}}, \quad c_3 = \sqrt{\frac{C_{44}^0}{\rho_0}}, \]

(24)

\[ C_{44c}^r = C_{44e}^0 + \left[ \eta_1^0 \right]^2 \]

The rest of the proof is similar to that of (11) and so is omitted. From (14), with a proof similar to the one of (13), we obtain the following result. Regarding the oscillation of (1), we refer to the reader the books [1, 2] and the articles [12, 29, 31] when \( p(t) > 1 \). In this paper, we try to obtain some new oscillation criteria for (1). The paper is organized as follows: In the next section, under the cases (2) or (3), we will utilize the Riccati transformation technique to obtain some sufficient conditions for the oscillation of (1). We shall give several examples to illustrate the main results. On this basis, we give some remarks to compare our results with those in the literature. In the sequel, for the sake of convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large \( t \) which can be further represented as:

\[ 0 \leq \eta_1^0 \leq 21, 11, () \]

(28)

Thus, \( \bar{g}(r, \omega) \) must be determined, that is:

\[ \bar{g}(r, \omega) = \int_{-\infty}^{\infty} e^{i\omega t} \bar{g}(r, t) dt \]

(30)

Put (28) into (30) to obtain:

\[ \bar{g}(r, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \frac{e^{i\omega t} dt}{\sqrt{t^2 - r^2}} \]

(31)

As per (33), \( \bar{g}(r, \omega) \) of Equation (32) can be obtained:

\[ \bar{g}(r, \omega) = \frac{i}{4} H_0^1(\omega r) \]

(34)

The function component defined for Fourier transform can be obtained in the following equation (35)-(40).

\[ G_a(r, \omega) = \frac{i}{4\rho_0\omega} \{ \theta_a \beta^2 H_0^1(\beta r) \]

\[ - \frac{\partial}{\partial y_i} [H_0^1(qr)]_{\beta^2} + m_i \beta \epsilon_0^2 H_0^1(\beta_0 r) \}

(35)

\[ g(r, \omega) = \frac{1}{2\pi \eta_1^0} \ln r + \frac{i}{4\rho_0\omega} \left( \epsilon_0^2 \right)^2 H_0^1(\beta_0 r) \]

In which:

\[ f(qr)_{\beta^2} = f(\alpha r) - f(\beta r), \quad r = |y| \]

(36)

\[ G_{ik}(r, t) = \frac{\theta_{ik}}{C_{66}} \bar{g}_i(r, t) + \frac{1}{\rho_0} \frac{\partial^2}{\partial y_i \partial y_k} \]

(37)
In which, $g_i(r, t)$ represents output of Equation (38)

$$g_i(r, t) = \frac{\delta(t)}{2\pi \eta_1} \ln r + \frac{(e_i^0)^2}{(\eta_1^0)^2} C_i \bar{g}_i(r, t)$$

IV. CONCLUSION

In this paper, the author studies the numerical analysis of nonlinear neutral functional differential equations. Functional differential equations (FDEs) arise widely in physics, biology, engineering, medical science, economics and so on. By this simulation, to get the solution we only need to calculate the integral which is much easier than other methods. Furthermore, we must point out that the fundamental solution can be treated as adjusted transitional function by the relationship with the transitional probability density function of diffusion. It is meaningful to study the theory and application of numerical methods for FDEs. The result shows gets B-stability, B-consistency and B-convergence results of numerical methods which are more general and deeper than the existing related traditional method.

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