Fibrations Theory and Indexed Co-Inductive Data Types

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Abstract — In this paper we consider basic logical structures of indexed co-inductive data type (ICDT, in short) over a fibration such as truth and quotient functor. We present an indexed fibration, then propose the relation fibration and equation functor of indexed fibration, and construct co-recursive operation to analyze semantic behavior of ICDT and its co-inductive rule by co-algebra category and functor lifting. Finally, we briefly introduce the applications of Fibrations theory to ICDT by example.

Keywords - ICDT; Fibrations theory; Lifting; Semantic behavior; Co-algebra category

I. INTRODUCTION

ICDT is a co-inductive data type, whose ability of semantic computing is stronger; it can deal with more complex data structures. Traditional methods including category theory and co-algebra make model in local Cartesian closed category, ICDT and relation category describing its semantic behavior coexist the same category, which results in the functor and its lifting are identical, so there exist some limits in analyzing semantic behavior.

Fibrations is well explained in Wikipedia as: In topology, a branch of mathematics, a fibration is a generalization of the notion of a fiber bundle. A fiber bundle makes precise the idea of one topological space (called a fiber) being "parameterized" by another topological space (called a base). A fibration is like a fiber bundle, except that the fibers need not be the same space, nor even homeomorphic; rather, they are just homotopy equivalent. Weak fibrations discard even this equivalence for a more technical property.

Fibration theory is an emerging trend in computer science, whose application to ICDT can fuse traditional methods effectively, and provides a mathematical framework to analyze semantic behavior. Furthermore, the relation describing semantic behavior of ICDT doesn’t limit to morphism, but lifts to objects in total category. More importantly, ICDT and relation category describing its semantic behavior do not coexist in the same category, but constructing functor lifted in total category to depict abstractly semantic computing and program logics of ICDT.

In the work of [1], Hermida and Jacobs did lots of works. A functor (see Wikipedia) is a type of mapping between categories arising in category theory. Functors can be thought of as homomorphisms between categories. In the category of small categories, functors can be thought of more generally as morphisms.

In this paper, we apply Fibrations theory to research ICDT, by taking ICDT to be object set in base category, taking their semantic behavior to be object set in total category firstly; then we establish the responsible relation in program logic directly between ICDT and it semantic behavior by equation functor and quotient functor, and construct co-recursive operation of ICDT to describe abstractly co-inductive rule with universality by the tools of endo-functor in base category and their lifting equation-preserving in total category.

II. PRELIMINARIES

All research objects in this paper are based on small category [2], more details about Fibrations theory can be found in [2-4]. For category \( \mathcal{C} \), let \( \mathcal{O}_\mathcal{C} \) to be set of objects and \( \text{Mor} \mathcal{C} \) to be set of morphisms.

For \( f: C \rightarrow D \in \text{Mor} \mathcal{B} \), \( P(Y) = D \), we write \( f^Y \) for Cartesian arrow of \( f \) and \( Y \). Let \( P: \mathcal{T} \rightarrow \mathcal{B} \) be a fibration, we call \( \mathcal{B} \) the base category and \( \mathcal{T} \) the total category of \( P \). For an object \( C \) in \( \text{Obj} \mathcal{B} \), \( \exists X \in \text{Obj} \mathcal{T}, k \in \text{Mor} \mathcal{T} \), if it satisfies \( P(X) = C \) and \( P(k) = \text{id}_C \), then the subcategory \( \mathcal{T}_C \) constituted by \( X \) and \( k \) is called to be a fiber over \( C \). We call \( f^X: \mathcal{T}_C \rightarrow \mathcal{T} \), a re-indexed functor, and \( f^X: \mathcal{T}_C \rightarrow \mathcal{T} \) is an...
opposite re-indexed functor. Similar for \( f^\ast_{\alpha} \), we write \( f^\ast_{\alpha} \rightarrow \) for the opposite Cartesian arrow \( u \) of \( f \) and \( X \).

III. ICDT AND ITS SEMANTIC BEHAVIOR

A. Indexed Fibration

**Definition 1.** Let \( P : \mathcal{T} \rightarrow \mathfrak{B} \) be a fibration between \( \mathcal{T} \) and \( \mathfrak{B} \), and functor \( T_P : \mathfrak{B} \rightarrow \mathcal{T} \) maps \( \forall C \in \mathfrak{B} \) to a terminal object in fiber \( \mathcal{T}_C \). Then \( T_P \) is a truth functor of fibration \( P \).

If \( T_P \) has a right ad joint functor \( \{ \} \), then \( \{ \} \) is a comprehension functor of \( P \).

**Definition 2.** Let \( P : \mathcal{T} \rightarrow \mathfrak{B} \) be a fibration between \( \mathcal{T} \) and \( \mathfrak{B} \), and its base category \( \mathfrak{B} \) has products. Let \( \Delta : \mathfrak{B} \rightarrow \mathfrak{B} \) be a diagonal endo-functor on \( \mathfrak{B} \), it maps \( \forall C \in \mathfrak{B} \) to product object \( C \times C \). The pullback of \( P \) along \( \Delta \) constructs fibration \( Rel_P : Rel(\mathcal{T}) \rightarrow \mathfrak{B} \), and \( Rel(P) \) is a relation fibration of \( P \).

Pullback-preserving property of Definition 2 ensures that fiber \( Rel(\mathcal{T}) \rightarrow C \) on \( Rel(P) \) is isomorphism up to fiber \( \mathcal{T}_{C \times C} \) above \( C \times C \) on \( P \), i.e., \( Rel(\mathcal{T}) \rightarrow C \). The process of constructing a new fibration by given fibration is change of base. For example, we construct \( Rel(P) \) by change of base in Definition 2 from \( P \). Change of base preserves structure, such as preserves fibered terminal objects [5], that is, if \( P \) has truth functor \( T_P \), then \( Rel(P) \) has truth functor \( T_{Rel(I)}(P) \), and \( T_{Rel(P)}(C) = T_P(C \times C) \).

**Definition 3.** Let \( P : \mathcal{T} \rightarrow \mathfrak{B} \) be a bifibration satisfying Beck-Chevalley condition [1], base category \( \mathfrak{B} \) has products, and \( T_P \) be the truth functor of \( P \). For \( \forall C \in \mathfrak{B} \), the action \( \delta_C : C \rightarrow C \times C \) of natural transformation \( \delta : Id_y \rightarrow \Delta \) at \( C \) extends an opposite re-indexed functor \( \delta^\ast \). \( Eq_p : \mathfrak{B} \rightarrow Rel(\mathcal{T}) \) is an equation functor of \( P \), and \( Eq_p = \delta^\ast \circ T_P \).

Truth functor \( T_P \) maps \( C \) to the terminal object \( T_P(C) \) in \( \mathcal{T}_C \), by Definition 2 \( Rel(P) \) is the change of base of \( P \) along \( \Delta \). If \( P \) has fibered terminal object, then \( Rel(P) \) also has fibered terminal object. The opposite re-indexed functor \( \delta^\ast \) in Definition 3 maps \( T_P(C) \) to \( \delta(T_P(C)) \), and \( \delta(T_P(C)) \in Obj(\mathcal{T}_{C \times C}) \), equation functor \( Eq_p \) of \( P \) maps \( \forall f \in \mathcal{M} \) to the unique morphism above \( f \times f \) determined by \( \delta_f \) and \( (\delta_C)^{\ast \times \ast} \). The meaning of equation functor is that same parameter has same results [1].

**Theorem 1.** Let \( P : \mathcal{T} \rightarrow \mathfrak{B} \) be a fibration or bifibration between \( \mathcal{T} \) and \( \mathfrak{B} \), and \( T_P : \mathfrak{B} \rightarrow \mathcal{T} \) be a truth functor of \( P \). There \( \exists I \in \mathcal{E} \), where \( I \) is a discrete indexed object in base category \( \mathfrak{B} \). Let indexed functor \( P/I : \mathcal{T} \rightarrow T_P(I) \rightarrow \mathfrak{B} \) be \( P/I(u) = P(u) : P(Y) \rightarrow I \in \mathfrak{B} \) for \( \forall u : Y \rightarrow T_P(I) \in \mathfrak{B} \). Then the indexed functor \( P/I \) is also a fibration or bifibration.

**Proof.** For \( \forall f : C \rightarrow D \in \mathfrak{B} \), there exists a Cartesian arrow \( f^\ast : f^\ast(X) \rightarrow X \) above \( f \) on fibration \( P \). Then there exists a unique morphism \( v \in f^\ast \circ w \) and \( P(v) = v \circ h \). Let \( \alpha : D \rightarrow I \in \mathfrak{B} \), \( \beta : C \rightarrow I \in \mathfrak{B} \). Then \( \gamma : P(u) \rightarrow \alpha = P(Y) \rightarrow D \in \mathfrak{B} \), \( \delta : P(u) \rightarrow \beta = P(Y) \rightarrow C \in \mathfrak{B} \), which satisfies diagram commuting, that is, \( \gamma = f \circ \delta \). In total category \( \mathcal{T} \rightarrow T_P(I) \) on functor \( P/I \), there exist two objects \( s : X \rightarrow T_P(I) \in \mathfrak{B} \), \( t : f^\ast(X) \rightarrow T_P(I) \in \mathfrak{B} \) such that \( \alpha = \beta \circ \delta \). Then there exists a unique morphism \( k : u \rightarrow t \rightarrow Y \rightarrow f^\ast(X) \) satisfying diagram commuting \( g = f^\ast \circ k \). So \( f^\ast \) is a Cartesian arrow of \( f \) on functor \( P/I \), if \( P \) is a fibration, then the functor \( P/I \) is also a fibration.

Let \( m : Z \rightarrow T_P(I) \in \mathfrak{B} \) be an object in total category \( \mathcal{T} \rightarrow T_P(I) \). Then \( P/I(m) = \alpha \) by the definition of functor \( P/I \). Let \( f^\ast : Z \rightarrow f^\ast(Z) \) be an opposite Cartesian arrow of \( f \) on \( P \). The diagram commutes \( \alpha = \beta \circ f \) in slice category \( \mathcal{T}_I \), there exists an unique morphism \( n : f(Z) \rightarrow T_P(I) \) in total category \( \mathcal{T} \rightarrow T_P(I) \) on functor \( P/I \), which satisfies diagram commuting \( m = n \circ f^\ast \). So \( f^\ast \) is an opposite Cartesian arrow of \( f \) on functor \( P/I \) and \( P \) is an equivalent bifibration, then the indexed functor \( P/I \) is also an equivalent bifibration.

Therefore, if \( P \) is a fibration or bifibration, the single-indexed functor \( P/I \) is also a fibration or bifibration. □
fiber $\mathcal{T}$ above $C$ on $P$ is isomorphism up to the fiber $(\mathcal{T}/(T(I))_\alpha)$ above $\alpha$ on $P/I$[6].

B. Equation Functor of Indexed Fibration

For $\forall \alpha : C \to I \in \text{Obj}\mathcal{B}/I$, let two pullbacks of $\alpha$ along itself be $i$ and $j$ respectively. Then the product object $\alpha \times \alpha$ is $\alpha \circ i$ or $\alpha \circ j$, that is, product object $\mathcal{B}/I$ in slice category is determined by its pullbacks. Similarly to Definition 2, the following is definition of relation fibration and equation functor of indexed fibration $P/I$.

**Definition 4.** Let $P/I : \mathcal{T}/T_p(I) \to \mathcal{B}/I$ be an indexed fibration, base category $\mathcal{B}/I$ has products. Let $\Delta/I : \mathcal{B}/I \to \mathcal{B}/I$ be a diagonal endo-functor in slice category $\mathcal{B}/I$. Then $\Delta/I$ maps $\forall \alpha \in \mathcal{B}/I$ to product object $\alpha \times \alpha$. The pullback of $P/I$ along $\Delta/I$ constructs a fibration $F(I) : \mathcal{T}/(T_p(I)) \to \mathcal{B}/I$, $\mathcal{R}(P/I)$ is relation fibration of $P/I$.

**Definition 5.** Let $F : \mathcal{T} \to \mathcal{B}$ be a bifibration satisfying Beck-Chevalley condition between $\mathcal{T}$ and $\mathcal{B}$, $P$ has truth functor, and base category $\mathcal{B}$ has product. Let $T_{P/I}$ be a truth functor of indexed fibration $P/I$. Then

$$E_{P/I} = \delta(I) \circ T_{P/I} : \mathcal{B}/I \to \mathcal{R}(\mathcal{T}/(T_p(I)))$$

is an equation functor of $P/I$.

C. Quotient Functor and Its Lifting Equation-preserving

The truth functor $T_p : \mathcal{B} \to \mathcal{T}$ of fibration $P : \mathcal{T} \to \mathcal{B}$ is substituted by the equation functor $E_{P/I} : \mathcal{B} \to \mathcal{R}(\mathcal{T})$ of $P$, and $P$ is also substituted by its relation fibration $\mathcal{R}(P)$, then by Theorem 1 we construct a new fibration $\mathcal{R}(P)/I : \mathcal{T}/(T_p(I)) \to \mathcal{B}/I$. For $\forall R \in \text{Obj}\mathcal{R}(\mathcal{T})$, $\mathcal{R}(P)/I$ maps $\alpha : R \to E_{P/I}(I)$ to $\alpha' : QR \to I$, $\alpha'$ is the transpose [1] of $\alpha$ for ad joint functor $Q \downarrow E_{P/I}$.

**Definition 6.** Let the ad joint functor $\tau \downarrow \sigma : \mathcal{R}(\mathcal{T})/(T_p(I)) \to (\mathcal{R}(\mathcal{T})/E_{P/I}(I))$ satisfies diagram commuting, that is, $\mathcal{R}(P/I \circ \tau) = \mathcal{R}(P/I \circ \sigma)$ and $\mathcal{R}(P/I) = \mathcal{R}(P/I \circ \tau) \circ \sigma$. If $\mathcal{R}(P/I)$ has a right ad joint functor $E_{P/I}$ such that $E_{P/I}(\alpha) = \tau \circ E_{P/I}(\alpha)$, then $\mathcal{R}(P/I \circ \tau) = \mathcal{R}(P/I \circ \sigma)$, we call $\mathcal{R}(P/I \circ \tau)$ is quotient functor of indexed fibration $P/I$. Write $\mathcal{R}(P/I \circ \tau)$ for $Q_{P/I}$, and $Q_{P/I} \downarrow E_{P/I}$.

**Definition 7.** Let $F : \mathcal{T} \to \mathcal{B}$ be a bifibration satisfying Beck-Chevalley condition with truth functor $T_p$ between two small categories $\mathcal{T}$ and $\mathcal{B}$, base category $\mathcal{B}$ has products and pullbacks, and $P/I : \mathcal{T}/T_p(I) \to \mathcal{B}/I$ be an indexed fibration of $P$. We construct relation fibration $\mathcal{R}(P/I)$ of $P/I$, and equation functor $E_{P/I}$ and quotient functor $Q_{P/I}$ of $P/I$. Let $F$ be an endo-functor in base category $\mathcal{B}/I$ on relation fibration $\mathcal{R}(P/I)$.

If it satisfies diagram commuting $\mathcal{R}(P/I \circ F) = F \circ \mathcal{R}(P/I)$, and the following isomorphism holds, i.e., $E_{P/I} \circ F \equiv F^+ \circ E_{P/I}$ and $F \circ Q_{P/I} \equiv Q_{P/I} \circ F^+$, then $F^+$ is a lifting equation-preserving of $F$ about $\mathcal{R}(P/I)$ in total category $\mathcal{R}(\mathcal{T}/(T_p(I)))$.

D. Semantic Behavior of ICDT

For $\forall \alpha : C \to I \in \text{Obj}\mathcal{B}/I$, a $F$-coalggebra $(\alpha, r : \alpha \to F(\alpha))$ is constructed by the action of endo-functor $F$, $\alpha$ is called to be carrier. The $F$-coalggebra category is constituted by $F$-coalgebra and their morphisms, denoted as $\text{Coalg}_F$. ICDT $vF$ , as the carrier of terminal $F$-coalgebra, is maximal fixed point of functor $F$. The functor $F$ denotes syntax destructor of $vF$, and its morphism out observes semantic behaviors of $vF$ from outside during its syntax destructing. Applying equation functor $E_{P/I}$ of indexed fibration $P/I$ to map $F$-coalgebra $(\alpha, r)$ to a $F^+$-coalgebra $E_{P/I}(\alpha, r)$ = $(E_{P/I}(\alpha), E_{P/I}(r))$:

$$E_{P/I}(\alpha) = (E_{P/I}(\alpha), F(\alpha)) \equiv F^+(E_{P/I}(\alpha))$$

Write $\text{Coalg}(E_{P/I})$ for functor from $\text{Coalg}_F$ to $\text{Coalg}_{F^+}$, it maps objects and morphisms in base category $\mathcal{B}/I$ on relation fibration $\mathcal{R}(P/I)$ to those correspondingly in total category $\mathcal{R}(\mathcal{T}/(T_p(I)))$ by equation functor $E_{P/I}$ . Therefore, functor $\text{Coalg}(E_{P/I})$ establishes relationship between $\text{Coalg}_F$ and $\text{Coalg}_{F^+}$ further.

Let $(E_{P/I}(vF), \text{out}^+ : E_{P/I}(vF) \to F^+(E_{P/I}(vF))$ be a terminal $F^+$-coalgebra in total category $\mathcal{R}(\mathcal{T}/(T_p(I)))$ on relation fibration $\mathcal{R}(P/I)$. Then $\text{out}^+$ is a homomorphism image of $\text{out}$ by the action of functor $\text{Coalg}(E_{P/I})$, that is, $\text{Coalg}(E_{P/I})(\text{out}) = \text{out}^+$. Terminal property of terminal $F^+$-coalgebra ensures that $\text{out}^+$ is up to unique isomorphism, whose existence

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provides extremely convenience for analyzing accurately semantic behavior of ICDT.

Similar to \textit{Coalg}(Eq_{P_1}) , write \textit{Coalg}(Q_{P_1}) for the functor from \textit{Coalg}_{P_1} to \textit{Coalg}_{P} . Then we have \textit{Coalg}(Q_{P_1})-\textit{Coalg}(Eq_{P_1}) by the ad joint property of ad joint functor in [1]. For each \( F^+ \)-coalgebra \( (\alpha, q : \alpha \rightarrow F^+(\alpha)) \), \( \alpha : X \rightarrow T_p(I) \in \text{Obj Rel}(T / T_p(I)) \) , \( \text{Coalg}(Q_{P_1})(q) = \bigcup_b \text{Coalg}_{P_1}(F^+(\alpha)) \), that is, \( \text{Coalg}(Q_{P_1})(q) = Q_{P_1}(q) \) . Then \( Q_{P_1}(q) \) is a homomorphism image of \( q \) by the action of functor \textit{Coalg}(Eq_{P_1}) . If \( g : \alpha \rightarrow \text{Eq}_{P_1}(\alpha) \) is a \( F^+ \)-coalgebra morphism from \( q \) to \( Q_{P_1}(r) \) , then the \( F^+ \)-coalgebra morphism \( h : Q_{P_1}(q) \rightarrow \alpha \) from \( Q_{P_1}(q) \) to \( r \) is \( F^+ \)-coalgebra homomorphism above \( g \) .

The left ad joint \textit{Coalg}(Eq_{P_1}) of functor \textit{Coalg}(Eq_{P_1}) makes an intuitional inter-deducible relation between \( F \)-coalgebra \( Q_{P_1}(q) \) as its carrier and \( F^+ \)-coalgebra \( q \) as its carrier, which provides a succinct and consistent modeling method for formal description of ICDT, \( vF \) as carrier of terminal co-algebra.

IV. CO-INDUCTIVE RULE OF ICDT

For a fibration with equation and quotient functors, formal description of co-inductive rules and semantic behavior analysis of ICDT are coherent in [1].

Let \( P : T \rightarrow \mathbb{B} \) and \( P / I : T / T_p(I) \rightarrow \mathbb{B} / I \) also satisfy the requirements of Definition 7, \( F \) be an endo-functor in base category \( \mathbb{B} / I \) on relation fibration \( P / I \) of \( P / I \) , \( vF \) be the carrier of terminal \( F \)-coalgebra, and \( F \) has lifting equation-preserving \( F^+ \) . Then \( P / I \) has co-inductive rule, \( vF \) as ICDT.

This provides a sound basis determining if that \( F^+ \) applies terminal \( F \)-coalgebra to generate co-inductive rule on ICDT is valid, i.e., if indexed fibration \( P / I \) with equation functor and quotient functor analyzes semantic behavior of ICDT, its co-inductive rule based on terminal \( F \)-coalgebra is valid during the process of semantic behavior analysis in programming.

The following is co-inductive rule with universality presented and described abstractly in the framework of Fibrations theory for ICDT.

From the viewpoint of category theory, co-recursive computation of co-inductive data type stems from terminal coalgebra semantics [7]. For \( \forall \alpha : \mathbb{C} \rightarrow I \in \text{Obj} \mathbb{B} / I \) , and \( vF \in \text{Obj} \mathbb{B} / I \) , we apply \( F \) to construct co-recursive operation \( \text{unfold} : (\alpha \rightarrow F(\alpha)) \rightarrow \alpha \rightarrow vF \) of ICDT in base category \( \mathbb{B} / I \) .

For each \( F \)-coalgebra \( (\alpha, r : \alpha \rightarrow F(\alpha)) \) , \( \text{unfold} r \) maps \( r \) to the unique \( F \)-coalgebra morphism \( \text{unfold} r : \alpha \rightarrow vF \) from \( (\alpha, r) \) to the terminal \( F \)-coalgebra \( (vF, \text{out}) \) . The operation \( \text{unfold} \) derived from terminal coalgebra semantics is a co-recursive parameterized operation of ICDT, whose co-recursive computation has some favorable properties including correct semantics, flexible expansibility and succinct expression.

By Definition 7 we get \( \text{Eq}_{P_1}(F(\alpha)) \approx F^+(\text{Eq}_{P_1}(\alpha)) \), \( \text{Eq}_{P_1}(F(vF)) \approx F^+(\text{Eq}_{P_1}(vF)) \) , and equation functor \( \text{Eq}_{P_1} \) preserves terminal objects. Then \( \text{Eq}_{P_1}(vF) \) is the carrier of terminal \( F^+ \)-coalgebra.

Write \( vF^+ = \text{Eq}_{P_1}(vF) \) , and \( X = \text{Eq}_{P_1}(\alpha) \). Applying endo- functor \( F^+ \) constructs co-recursive operation \( \text{unfold} : (X \rightarrow F^+ (X)) \rightarrow X \rightarrow vF^+ \) of ICDT in total category \( \text{Rel}(T / T_p(I)) \) .

For any \( F^+ \)-coalgebra \( (X, q : X \rightarrow F^+ (X)) \) , \( \text{unfold} q \) maps \( q \) to the unique \( F^+ \)-coalgebra morphism \( \text{unfold} q : X \rightarrow vF^+ \) from \( (X, q) \) to the terminal \( F^+ \)-coalgebra \( (vF^+, \text{out}^+) \) .

\( \forall \alpha \in \text{Obj} \mathbb{B} / I \) , \( \exists X \in \text{Obj} \text{Rel}(T / T_p(I)) \) . The following is co-inductive rule of ICDT with universality.

\[ \text{Coind}_{\text{un}}^\omega : (X \rightarrow F^+ (X)) \rightarrow X \rightarrow \text{Eq}_{P_1}(vF) \]

If \( (X, q : X \rightarrow F^+ (X)) \) is a \( F^+ \)-coalgebra over \( F \)-coalgebra \( (\alpha, r : \alpha \rightarrow F(\alpha)) \) , then \( \text{Coind}_{\text{un}}^\omega X q \) is \( F^+ \)-coalgebra homomorphism over \( \text{unfold} r \).

\textbf{Example 1}. The element type of stream is designated by \( index I \) , such as \textit{natural number Nat} and \textit{character Char} , \( \forall i \in \text{Obj} \mathbb{B} \) . For any stream \( \alpha : S \rightarrow I \in \text{Obj} \mathbb{B} / I \) , we can construct an endo-functor \( F^+ : \alpha \rightarrow I \times \alpha \) on \( \mathbb{B} / I \) , where \( \text{head} \alpha \rightarrow I \) is head function of stream, and \( \text{tail} \alpha : \alpha \rightarrow \alpha \) is tail function erased head element. For any stream property \( R \in \text{Obj} \text{Rel}(T / T_p(I)) \) in total category \( \text{Rel}(T / T_p(I)) \) on relation fibration \( \text{Rel}(P / I) \) of indexed fibration \( P / I \) , such as \textit{bi-simulation}, then for another stream object \( \beta : S^+ \rightarrow I \in \mathbb{B} / I \) , a co-induction of \( \alpha \) and \( \beta \) on \textit{bi-simulation} property \( R \) is established:

\[ \text{Object } R \text{ is relation of bi-simulation between two streams } \alpha \text{ and } \beta \text{, iff } \forall (\alpha, \beta) \in R \text{, } \text{head}(\alpha) = \text{head}(\beta) \text{, and } (\text{tail}(\alpha),\text{tail}(\beta)) \in R \].
Let the stream $\text{Stream}(I)$ be carrier $vF$ of terminal $F$ -coalgebra $(\nu F, ou : vF \rightarrow F(\nu F))$ in base category $\mathcal{B} / / I$. For each $F$ -coalgebra $(\alpha : r : \alpha \rightarrow F(\alpha))$, it is lifted to a $F^+ -coalgebra \langle X, q : X \rightarrow F^+(X) \rangle$ by relation fibration $\text{Rel}(P / I)$, which satisfies diagram commuting, i.e., $F \circ \text{Rel}(P / I)(X) = \text{Rel}(P / I) \circ F^+(X)$. The terminality of terminal $F$ -coalgebra defines a co-recursive operation unfold $r$ on $\text{Stream}(I)$, which executes the judgment of ICDT $\text{Stream}(I)$; another co-recursive operation by the terminality of terminal $F^+ -coalgebra$ describes semantic behavior of $\text{Stream}(I)$. Iterating each property $R$ in total category $\text{Rel}(\tau / P(I))$ on relation fibration $\text{Rel}(P / I), R \in \text{Obj} \text{Rel}(\tau / P(I))$, we can obtain the semantic set $\{R(X, X) | X = \text{Eq}_{P(I)}(\alpha), \forall \alpha \in \text{Obj} \mathcal{B} / / I\}$, which depicts behaviors of $\text{Stream}(I)$.

In Example 1, unfold $r$ describes mapping relationship between streams $\alpha$ and its semantic behavior intuitively. The existence of unfold $r$ provides a valid way of homomorphism from coalgebra to its terminal coalgebra. To define function unfold $r : \alpha \rightarrow \text{Stream}(I)$, we only need to construct corresponding operation $r$ on $\alpha$, and let $(\alpha : r : \alpha \rightarrow F(\alpha))$ be a $F$ -coalgebra, $F(\alpha) = I \times \alpha$; meanwhile, the uniqueness of unfold $r$ can prove two homomorphism are equivalent further. Therefore, we have co-induction proof principle, that is, to prove $m, n : \alpha \rightarrow \text{Stream}(I)$ is equivalent, we only need to prove $m$ and $n$ also is homomorphism from the same coalgebra $(\alpha, r)$ to its terminal $F$ -coalgebra ($\text{Stream}(I), ou : \text{Stream}(I) \rightarrow F(\text{Stream}(I))$), namely, $m$ and $n$ also equal unfold $r$.

V. RELATED WORKS

From the situation of literature retrieval, Hagino is maybe the earliest scholar who researched relationship between inductive and co-inductive data type systematically by di-algebras structure in [8], whose works laid the foundation, but there exist drawbacks in polymorphism type system, co-induction application of programming and so on. Nogueira studied polymorphism type system by bi-algebra [9]. Poll extended the works of Hagino based on sub-type and inheritance [10]. Greiner brought co-inductive principle in program languages, deeply discussed co-inductive data type in programming [11], the works above solved questions aforementioned to some extent. Hermida and Jacobs proved co-inductive rule of terminal co-algebra with quotient type in [1], Ghani etc. broke the limitation of polynomial functor based on [1], extended their works to general functor [6].

Existing researches mainly focus on co-inductive data type, but studying for ICDT is still in a preliminary stage currently. There exist plenty of unsolved problems in semantic computing and program logic. For example, analyzing semantic behavior and depicting co-inductive rule, especially the latter is mostly generated automatically, which is lack of solid math foundations and succinct formal description.

This paper explores semantic behavior of ICDT systematically and deeply by Fibrations theory. Compared with traditional methods including algebra and category theory, the advantages of this paper are as follows:

- It analyzes semantic behavior of ICDT succinctly by Fibrations theory, improves abilities of processing and proving of program languages for semantic behavior of ICDT;
- It also presents and describes abstractly co-inductive rule of ICDT in the framework of Fibrations theory, not depending particular computing circumstance;
- Those works provide solid math foundations and succinct and uniform descriptive ways for semantic computation and program logic of program languages.

Applying Fibrations theory to research ICDT in programming is extension and deepening of traditional methods for co-inductive data type in the level of category theory. After coalgebra methods appeared especially, organic combination of some dual categorical concepts, such as Cartesian arrow and opposite Cartesian arrow, fibration and op-fibration and so on, present powerful vitality for Fibrations theory to explore ICDT in program languages. It also has wide development prospect in foundation theory of computer science and engineering practice.

In the meantime, applying Fibrations theory to study ICDT in programming is not pure mathematical exploring; but from the application view of program languages, combining Fibrations theory with latest researching fruits of oriented object program languages, coalgebra specifications and formal semantics, it engages in basic research to some kernel problems of ICDT in program language systematically and deeply, such as categorical properties of core concepts, semantic behavior and specification describing.

VI. CONCLUSIONS

Fibrations theory integrates traditional thinking of program languages, whose characteristic thinking and researching method of high abstraction, flexible extension and succinct description produce vigorous and profound impact in programming, and promotes applications of category theory in computer science extremely.

But from the status of documents retrieval nowadays, there are a few scholars working on Fibrations theory internationally, and small amount of literatures applied Fibrations theory to computer science relatively; especially
the literatures exploring systematically and deepen program languages and its formal semantics are even less, and in China we haven’t find out other scholars are going in for Fibrations theory and its applications in computer science nowadays.

Fibrations theory has particular advantages on solving description of abstract problems; meanwhile, it also has wide application prospective in theoretical computer science. This paper studies semantic behavior analysis and co-inductive rule description of ICDT preliminarily, we hope our works can raise concerns of other scholars especially in China for Fibrations theory, and then promote applications and developments of Fibrations theory and its application in computer science altogether.

This paper modelled on slice category $\mathbb{B}/I$, dealt with well semantic behavior analysis of single-sorted ICDT which takes $I$ as its index. But the index $I$ only points at some particular single-sorted ICDT; it is difficult to manage more complex many-sorted ICDT such as mutual recursive. Therefore, extending discrete index object $I$ of single-sorted indexed fibration to indexed category $\mathcal{C}$ to construct many-sorted fibration, representing many-sorted ICDT in $\mathbb{B}$ by index set $\mathcal{O}_I \mathcal{C}$, modelling semantic behavior of many-sorted ICDT in indexed category $\mathcal{C}$ by fibration $G: \mathbb{B} \to \mathcal{C}$, selecting different program logics aiming at different indexes are our future works. At the same time, our future works will discuss completeness, soundness and consistence of formal system constructed by ICDT and their co-inductive rules preliminarily. Furthermore, expanding our works of ICDT to 2-category by Fibrations theory, exploring deeply math structure and categorical property of semantic computing and program logic in high-order category are still our future works.

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