A Novel Droop-Logistic Model for Microorganism Population Studies

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Abstract - In this work the Droop model and logistic model are combined to form another mathematical model for a microorganism population that is named the Droop-Logistic model. The equation of the organism growth of this model is from the logistic model, and the growth rate is from the Droop model. Our new model is shown to have a unique solution on an open set by the Lipschitz condition. By analyzing local stability, the condition for having maximum cell numbers and the condition for being stable from the balancing of the surrounding nutrient and the intracellular quota are determined. Numerical examples are given three values of dilution rate. It was found that when the dilution rate satisfies the condition of maximum growth, i.e. it is less than the maximum growth rate, then the cell number will reach its maximum at the stationary time. If the dilution rate is greater than the maximum growth rate, then the cell number will decrease to zero. Lastly, if the dilution rate is zero and the maximum growth condition is satisfied, then the cell number will tend to the maximum value as well.

Keywords - Droop model; logistic model; population; nutrient

I. INTRODUCTION

A mathematical model is a tool to envision phenomena logically. A differential equation is a type of the model which is commonly used in microorganism research, for instance, microalgae and bacteria [3, 10, 22]. This field of research has a high impact on innovative works because of the products of the cells [16, 19, 21] and to environmental and watery ecological studies [9, 12, 18, 26, 29]. Before choosing the subtype of differential equation for a topic, the assumption of the organism growth must be stated.

The growth of a microorganism depends on two main types of factor. Those are outside-cell factors and inside-cell factors. The factors outside the cells are, for example, surrounding nutrients and environmental factors such as temperature, light, living space, other organisms, and other surrounding factors. Inside the cell, the metabolism process according to its gene effects to the growth. Adjacent nutrients are passed into the cell by its enzyme and cell wall. Then they are used to synthesize ATP that is the energy for cell activities [25], and used to store energy in the form of starch, lipids, etc. [8]. This storage is for non-light time and reproductive time [7]. However, the cell number is limited by the volume of living space because the resources, e.g., light, decreases when the cells are dense. These facts are the rules to make the assumptions of a model as well as the Droop model and Logistic model.

Those two models have differently prominent points. For the Droop model, the change of growth of the organisms can be described and predicted by changing the substrate concentration made by those organisms and the dilution. Its growth rate depends on the cell quota for the nutrient, which is the nutrient usage, and the change of cell quota depends on the focused nutrient in the supply [3]. For the logistic model, its growth rate is a constant and the growth is bounded by the environmental term [2]. This term is a constant and it can be an agent of other factors that we do not mention in equations. Moreover, the growth curves of the organism from both models are similar because the equations of the Droop model can be rewritten in the form of the logistic model when the cell quota is a constant [17]. Thus, the combination of these distinctive points will be a developed model that is suitable for cell number data that depend on the nutrient usage of the population and limited by other factors.

In this research, we combine the Droop model and the logistic model together to be another model that we call the Droop-Logistic model. The new model is focused on only a single substrate. We analyze the existence and uniqueness, and local stability of solutions. Lastly, examples of numerical results are given.
II. DROOP-LOGISTIC MODEL

In the situation that each cell of the microorganism has the same conditions in both outer and inner sides of the cell, the Droop model has demonstrated its efficiency to describe and predict the growth of biomass concentration with substrate removal [4, 5, 13, 28]. By letting the intracellular quota \( Q \) be the agent of nutrient usage inside the cell, the Droop model gives rise to three equations as follows:

\[
\begin{align*}
\dot{X} &= (\mu(Q) - D) \cdot X, \\
\dot{S} &= -\rho(S) \cdot X + D \cdot (S_{in} - S), \\
\dot{Q} &= \rho(S) \cdot X - \mu(Q) \cdot Q,
\end{align*}
\]  

where \( X \) is the biomass concentration (mgCell/L), \( S \) is the substrate concentration (mgNutrient/L), \( Q \) is the intracellular quota (gNutrient/gCell), \( \rho(S) = \rho_m \left( \frac{S}{S+K_S} \right) \) is the uptake rate, \( \rho_m \) is the maximum absorption rate ((mgNutrient/mgCell)/day), and \( K_S \) is the half saturation constant. \( \mu(Q) = \mu_m \left( 1 - \frac{Q_0}{Q} \right) \) is the specific growth rate, \( \mu_m \) is the maximum growth rate (1/day), \( Q_0 \) is the minimum cell quota (gNutrient/gCell), \( D = \) (inlet flow rate)/(culture volume) is the dilution rate (1/day), and \( S_{in} \) is the substrate concentration at the initial time.

From Fig. 1, both biomass concentration and cell density can affect the growth of the organisms. Biomass concentration is needed to calculate cell product and nutrient removal [6, 15, 20, 23], whereas cell density is sufficient for observing the growth [1, 11, 14, 30]. An acceptable equation for fitting a curve to the observed cell number is the logistic model [28]:

\[
\dot{X} = r \cdot X(t) \cdot \left( 1 - \frac{X(t)}{K} \right),
\]

where \( X(t) \) is the population function, \( r \) is the growth rate, and \( K \) is the carrying capacity for the population and culture.

To combine the Droop model and logistic model, we use the growth equation of the logistic model (2), and then we change its constant growth rate \( r \) to be \( \mu(Q) - D \) from the Droop model (1). Now, the changed system is

\[
\begin{align*}
\dot{X} &= (\mu(Q) - D) \cdot X \cdot \left( 1 - \frac{X(t)}{X_k} \right), \\
\dot{S} &= -\rho(S) \cdot X + D \cdot (S_{in} - S), \\
\dot{Q} &= \rho(S) \cdot X - \mu(Q) \cdot Q,
\end{align*}
\]

where \( X_k \) is the maximum cell density, and the new units are shown in Table I.

<table>
<thead>
<tr>
<th>Variables and Parameters</th>
<th>for cell density</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>cell/L</td>
</tr>
<tr>
<td>( S )</td>
<td>mg/L</td>
</tr>
<tr>
<td>( Q )</td>
<td>mg/cell</td>
</tr>
<tr>
<td>( \mu_m )</td>
<td>1/day</td>
</tr>
<tr>
<td>( Q_0 )</td>
<td>mg/cell</td>
</tr>
<tr>
<td>( K_S )</td>
<td>mg/L</td>
</tr>
<tr>
<td>( \rho_m )</td>
<td>(mg/cell)/day</td>
</tr>
<tr>
<td>( D )</td>
<td>1/day</td>
</tr>
</tbody>
</table>

The system (3) is different from the Droop model in only the equation for \( \dot{X} \). Therefore an assumption used in developing the Droop model is violated in the new system. That assumption is the law of conservation of nutrients [18]:

\[
S(t) = S_0 - Q(t) \cdot X(t).
\]

It is obvious that (1) agrees with (4), but (3) does not. However, this law of nutrient must hold in our expected model. Then we must change either the \( \dot{S} \) equation or the \( \dot{Q} \) equation of (3) to make the assumption true. If we
change the $\dot{Q}$ equation, mixing the $\dot{S}$ and the $\dot{X}$ equations, and then substitute them into (4), we will have

$$\dot{Q} = \rho(S) \cdot X - Q \cdot \mu(Q) \cdot X \cdot \left(1 - \frac{X}{X_k}\right).$$

$$\dot{S} = -\left(\rho(S) \cdot \frac{\mu(Q) \cdot X}{X_k} + D \cdot Q \cdot X \cdot 1 - \frac{X}{X_k}\right) \cdot X,$$

Then $\dot{Q} = \rho(S) \cdot Q - \mu(Q) \cdot Q$.

These two equations give different results when $X(t) = 0$. So it does not make sense to use this formula for $\dot{Q}$. Thus we change the formula of $\dot{S}$ instead. Then, the equations of our final system that we name the Droop-Logistic model are as follows:

$$\dot{X} = (\mu(Q) - D) \cdot X \cdot \left(1 - \frac{X}{X_k}\right),$$

$$\dot{S} = \left(-\rho(S) \cdot \frac{\mu(Q) \cdot X}{X_k} + D \cdot Q \cdot X \cdot 1 - \frac{X}{X_k}\right) \cdot X,$$

$$\dot{Q} = \rho(S) \cdot X - \mu(Q) \cdot Q.$$

### III. MODEL ANALYSIS

#### A. Existence and Uniqueness

Firstly, we check that our model has the unique solution on the open set $E = \{(X,S,Q) \mid X > 0, S > 0, Q > 0\} \subset \mathbb{R}^3$ by using the Lipschitz condition.

**Theorem 1** The Droop-Logistic model, system (5), satisfies the Lipschitz condition on $E$.

**Proof.** For all $Y_1 = \begin{bmatrix} X_1 \\ S_1 \\ Q_1 \end{bmatrix}$ and $Y_2 = \begin{bmatrix} X_2 \\ S_2 \\ Q_2 \end{bmatrix}$ in $E$, there is a positive constant $\Lambda = \text{Max} \{C_1, C_2, C_3\}$ such that

$$\Lambda \|Y_1 - Y_2\| = \text{Max} \{C_1, C_2, C_3\} \sqrt{(X_1 - X_2)^2 + (S_1 - S_2)^2 + (Q_1 - Q_2)^2} \geq \sqrt{C_1^2 (X_1 - X_2)^2 + C_2^2 (S_1 - S_2)^2 + C_3^2 (Q_1 - Q_2)^2}.$$

Next, we will examine the stability of the equilibrium points of our model.

#### B. Equilibrium Points

To find the equilibrium points of system (5), we let $f(Y) = 0$. Then we find that the equilibrium points are in two lines on $E$:

$$Y_1^* = \begin{bmatrix} X_k \cdot S \\ \frac{S}{S + K_s} \end{bmatrix},$$

$$Y_2^* = \begin{bmatrix} X \cdot S \end{bmatrix},$$

where

$$S^* = \begin{bmatrix} \frac{\mu_m Q_0}{\mu_m - D} \\ \frac{\mu_m}{\mu_m - D} \end{bmatrix}.$$

The first case of equilibrium points, $Y_1^*$, is the case that the population reaches its maximum $X_k$ for that culture. In the second case, $Y_2^*$, the system stops changing by the balancing between the cell quota $Q$ and the substrate concentration $S$.

Before choosing the method to analyze the stability of these equilibrium lines, we must check whether our equilibrium points in those lines are hyperbolic or not. We find the linearization $Df(Y)$ of our system. Then, if
The equilibrium points of the Droop-Logistic model (5) are non-hyperbolic.

Proof. There is a zero real-part eigenvalue of \( \frac{\partial f}{\partial Y} \) as shown below, where \( \lambda \) is an eigenvalue.

\[
\lambda_1 = \frac{-\mu_m + \frac{\mu_m q_0}{\rho_m \left( \frac{S}{S + K_S} \right)} + q_0}{\left( \frac{S}{S + K_S} \right)} - \frac{\mu_m k_s X_k}{\left( \frac{S}{S + K_S} + k_s \right)^2 (\mu_m - D)}
\]

\[
\lambda_2 = \frac{-\mu_m + \frac{\mu_m q_0}{\rho_m \left( \frac{S}{S + K_S} \right)} + q_0}{\left( \frac{S}{S + K_S} \right)} - \frac{\mu_m k_s X_k}{\left( \frac{S}{S + K_S} + k_s \right)^2 (\mu_m - D)}
\]

\[
\lambda_3 = 0
\]

where

\[
A = \left( \mu_m + \frac{k s m X_k}{(S + K_S)^2} \right) - 4 \left( \frac{\mu_m k s X_k}{(S + K_S)^2 (\mu_m - D)} \right) X_k \left( 1 - \frac{X}{X_k} \right)
\]

Since all equilibrium points are non-hyperbolic, we will find Lyapunov functions to examine their stability.

C. Local Stability

One of famous local stability theories is Lyapunov Stability. The stability of an equilibrium point is explored by a real valued function as known as a Lyapunov function, \( L(Y) \), which has two properties: \( L(Y^*) = 0 \) when \( Y^* \) is the equilibrium point, and \( L(Y) > 0 \) for all \( Y \neq Y^* \). The equilibrium point is stable if \( \dot{L}(Y) \leq 0 \) for all \( Y \) and it will be asymptotically stable when \( \dot{L}(Y) < 0 \) for all \( Y \neq Y^* \) [24].

Theorem 2 X of the Droop-Logistic model, system (5), will reach its maximum \( X_k \) and stay at this value, if

\[
\mu_m > \mu_m \left( 1 - \frac{Q_0}{Q} \right) \geq D,
\]

for all \( Q \).

Proof. By defining a Lyapunov function \( L(Y) = X_k - X \), the equilibrium line \( Y^* = \left( \frac{X_k - S + \frac{S}{S + K_S}}{S + K_S} \right) + Q_0 \) is asymptotically stable when that inequality holds.

Theorem 3 X of the Droop-Logistic model, system (5), will be in a steady state because of the balance of \( S \) and \( Q \) if

\[
\rho_m \left( \frac{K_S}{S + K_S} \right) - \mu_m \left( \frac{Q_0 - Q}{S + K_S} \right) \leq 0
\]

for all \( X, S, Q \).

Proof. This condition holds by defining a Lyapunov function \( L_2(Y) = \frac{\rho_m \left( S + K_S \right)}{\mu_m (S + K_S)} + q_0 - Q \).

IV. NUMERICAL SOLUTIONS

This section will give examples of \( Y_1^* \) and \( Y_2^* \) for \( D \neq 0 \) and \( D = 0 \). If we want to set \( S \) to be a constant by refilling the substrate to the system, we can calculate it by setting \( S = 0 \). That is, at time \( t = 0 \),

\[
\dot{S}(0) = \left( \frac{\rho_m (Q_0 - Q)}{X_k} \right) X_0 + \left( \frac{\rho_m (Q_0 - Q)}{X_k} \right) X_0 = 0
\]

then

\[
D = \frac{\rho_m S_0}{Q_0 (S_0 + K_S)} \left( 1 - \frac{X_0}{X_k} \right)
\]

We let \( X_0 = 4.93 \times 10^6 \), \( S_0 = 12.25 \), \( Q_0 = 4.60 \times 10^{-7} \), \( X_k = 8.96 \times 10^6 \), \( K_S = 6.56 \), \( \rho_m = 5.7 \times 10^{-7} \), \( \mu_m = 0.6217 \), \( D_1 = 1.793 \geq \mu_m \) corresponding to (6), \( D_2 = 0.21085 < \mu_m \) following theorem 1, and \( D_3 = 0 \).

The solutions of (5) for the three cases of \( D \) are shown in Fig. 2-4.
V. CONCLUSION

We developed a mathematical model for population growth of microorganisms from two different models. The first model was the Droop model consisting of three equations of biomass, nutrient, and cell quota. The growth rate of the Droop model depends on nutrient usage of the cells. The second model is the Logistic model which is an equation of cell number. The growth of this equation is bounded by the maximum cell number for the environment, but its growth rate is a constant. Therefore we mixed the growth equation of the logistic model with the growth rate of the Droop model to be our growth equation of the organism. Then this equation and other two equations of the Droop model, for substrate and cell quota, were made the law of mass for the nutrient. After that we obtained (5) which we named the Droop-Logistic model. We examined that this model has a unique solution on an open subset of $\mathbb{R}^3$ and found that every equilibrium point of the model is non-hyperbolic. On that open subset, two equilibrium lines were found: one is of maximum cell density and the other is of the substrate and cell quota balancing. They are all stable under the conditions from the Lyapunov stability criteria. Therefore we got the condition for maximum growth of the organisms from an equilibrium line. Lastly, we gave examples of solutions from the Droop-Logistic model when focusing on three different dilution rates. The case of maximum growth and the case of entirely dead growth were found in our examples, but the studies of decreasing growth of the microorganism are rare. If there are available data sets of decreasing growth of the organisms, the Droop-logistic model could be an alternative tool to deal with them. Our model should be workable for many factors water source such as wastewater, because it has the logistic constant representing non-considering factors. And it can be developed easily because the equations are simple and based on the well-mixed assumption that is, every cell has the same conditions at every position at time $t$.

ACKNOWLEDGMENT

This research project is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.
REFERENCES


