EXACT STATIONARY SOLUTION TO TANDEM FLUID QUEUES

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Abstract: We consider a fluid system composed of multiple buffers in series. The first buffer receives fluid from a finite superposition of independent identical on-off sources. The active and silent periods of sources are exponentially distributed. The ith buffer releases fluid in the $(i+1)$th buffer. Assuming that the input rate of one source is greater than the service rate of the first buffer, the output process of each buffer can be modeled by an on-off source with the active period distributed as the busy period of an M/M/1 queue. For $i \geq 2$, the stationary content distribution of the ith buffer is obtained by the use of generating functions which are explicitly inverted.

***Keywords*:** Tandem fluid queues, output process, generating functions.

1 INTRODUCTION

In the area of telecommunication systems, stochastic fluid flow queues have been often used as burst scale models, e.g. for multiplexers. This approach ignores the discrete nature of the real information flow and treats it as a continuous stream. This information flow is modeled as an on-off type input process, which is temporarily buffered in a fluid queue, when it cannot be handled at once.
Here, we consider tandem fluid queues fed by a finite number of identical on-off sources. It is assumed that silent and active periods of the sources are independent and exponentially distributed. Tandem fluid queues are composed of consecutive infinite capacity buffers. The stationary behavior of the first buffer is explicitly derived in [Anick et al., 1982], using spectral decomposition arguments. As far as the other buffers are concerned, the output processes need to be characterized. In [Aalto, 1998] and [Boxma and Dumas, 1998], the authors consider a fluid queue driven by a superposition of on-off sources, with exponentially distributed silent periods and generally distributed active periods. Assuming that the input rate of one source is greater than the constant service rate of the buffer, they prove that the output process behaves as an on-off source with exponentially distributed silent periods and active periods distributed like the busy periods of a M/G/1 queue.

In this paper, we consider the stationary behavior of each buffer level in the tandem fluid queues, apart from the first one. Using results of [Aalto, 1998] and [Boxma and Dumas, 1998], the output processes look like on-off sources with active periods distributed as busy periods of an M/M/1 queue. This tandem of fluid queues has been studied in [Aalto and Scheinhardt, 2000], where the output processes have been considered as alternating renewal processes. The authors obtained the stationary fluid level distribution of each buffer in terms of a Bessel function integral. Here, we derive a new analytic expression of these distributions. By using the method developped in [Leguesdron et al, 1991] and [Barbot and Sericola, 2002], we write the solutions in terms of a matrix exponential and then via generating functions that are explicitly inverted. Nevertheless, as shown in the next section, we deal here with a more general setting than the one of [Barbot and Sericola, 2002].

2 MODEL FORMULATION

We consider M infinite capacity fluid queues in series. The first one is fed by the superposition of N independent identical on-off sources with exponentially distributed on-off periods with parameters μ and λ respectively. During the on period, a source emits fluid at a constant rate c_0. Thus, when n sources are emitting simultaneously, the input rate in the first buffer is nc_0. The first buffer empties in the second one at the rate c_1. For $i \geq 2$, the input of the ith buffer is the output from the $(i-1)$th buffer and its service rate is denoted by c_i. It is assumed that

$$0 < c_M < \ldots < c_1 < Nc_0$$

in order to avoid the trivial case where one or more buffers remain empty. Moreover, we make the restrictive assumption

$$c_0 \geq c_1 \tag{1}$$

which permits the output process of the first buffer to be simply derived. Under (1), when one or more sources are in state on, every buffer level increases. The traffic intensity of the first buffer is given by

$$\rho_1 = \frac{c_0 N\lambda}{c_1(\lambda + \mu)}.$$

For the purpose of stating easier the characterization of the output processes, we introduce the following definition.

Definition 2.1 *An on-off source is called an MM1(β,a,b,r) source if the off periods are exponentially distributed with rate β and the on periods are distributed as the busy periods of an M/M/1 queue with arrival rate a and service rate b. During the on periods, the source emits fluid at rate r.*

An MM1(β,a,b,r) source is controlled by an infinite birth and death process of which the non-zero entries of its infinitesimal generator A are given by

$$A_{0,0} = -\beta\,,\ A_{0,1} = \beta\,,$$

and for $j \geq 1$,

$$A_{j,j-1} = b\,,\ A_{j,j} = -(a+b)\,,\ A_{j,j+1} = a\,, \tag{2}$$

and the mean duration of busy periods is $1/(b-a)$ when $a < b$.
The following lemmas are proved in [Aalto, 1998] and [Boxma and Dumas, 1998].

Lemma 2.2 *In the stationary regime, the output process of the first buffer is equivalent to an MM1($N\lambda$,λ_1,μ_1,c_1) source where*

$$\lambda_1 = \lambda\left(N - \frac{c_1}{c_0}\right),$$
$$\mu_1 = \frac{\mu c_1}{c_0}.$$

Lemma 2.3 *In the stationary regime, the output process of a buffer with service rate c and fed by an MM1(β,a,b,r) source is equivalent to an MM1(β,a',b',c) source where*

$$a' = \frac{ac + \beta(r-c)}{r},$$
$$b' = \frac{bc}{r}.$$

Using Lemmas 2.2 and 2.3, it follows by induction that the output process of the ith buffer, for $1 \leq i \leq M$, is equivalent to an MM1($N\lambda$,λ_i,μ_i,c_i) source where

$$\lambda_i = \lambda\left(N - \frac{c_i}{c_0}\right)$$
$$\mu_i = \frac{\mu c_i}{c_0}.$$

For $i \geq 2$, the traffic intensity in the ith buffer is then given by

$$\rho_i = \frac{\dfrac{c_{i-1}}{\mu_{i-1} - \lambda_{i-1}}}{c_i\left(\dfrac{1}{N\lambda} + \dfrac{1}{\mu_{i-1} - \lambda_{i-1}}\right)} = \frac{c_0 N\lambda}{c_i(\lambda+\mu)}.$$

Consequently, we have

$$\rho_1 < \ldots < \rho_M$$

and the stability condition of the tandem fluid queues is $\rho_M < 1$.
Note that in [Barbot and Sericola, 2002], we considered a single fluid queue fed by a classical M/M/1 queue, which is, from our definition 2.1, a fluid queue fed by an MM1(a,a,b,r) source. Here we have to deal with MM1(β,a,b,r) sources, where $\beta \neq a$, which generalizes the results of [Barbot and Sericola, 2002].

3 A BUFFER FED BY AN MM1(β,a,b,r) SOURCE

We consider a single fluid buffer fed by an MM1(β,a,b,r) source. The service rate of the buffer is denoted by c, $c < r$. We derive an expression of the stationary buffer content distribution in terms of a series whose coefficients correspond to the successive powers of a *key matrix* G. The generating function of G is expressed as a function of the known generating function of a matrix T and is explicitly inverted.
The continuous time birth and death process associated with the MM1(β,a,b,r) source is denoted by $\{X_t,\ t \geq 0\}$ and its infinitesimal generator A is described by (2).

The drifts of that fluid queue represent the difference between the input and the service rates. Let d_j be the drift when X_t is in the state j. We thus have

$$d_0 = -c \text{ and } d_j = r - c \text{ for } j \geq 1.$$

The diagonal matrix containing these drifts is denoted by D. Since we are concerned by the stationary behavior of that fluid queue, we suppose that the stability condition is satisfied, that is

$$\rho_0 = \frac{r\beta}{c(b - a + \beta)} < 1. \quad (3)$$

The stationary state of the Markov chain $\{X_t, t \geq 0\}$ and the stationary amount of fluid in the buffer are denoted X and Q respectively. Let

$$F_j(x) = \Pr\{X = j, Q \leq x\}.$$

It is easy to see that for $j \geq 1$, we have $F_j(0) = 0$ and it has been shown in [Sericola and Tuffin, 1999] that $F_0(0) = 1 - \rho_0$. It is well-known, see e.g. [Mitra, 1988], that the functions F_j satisfy the following system of differential equations: for $x > 0$ and $j \geq 2$,

$$\begin{aligned} d_0 F_0'(x) &= -\beta F_0(x) + bF_1(x) \\ d_1 F_1'(x) &= \beta F_0(x) - (a+b)F_1(x) + bF_2(x) \\ d_j F_j'(x) &= aF_{j-1}(x) - (a+b)F_j(x) + bF_{j+1}(x) \end{aligned}$$

where $F_j'(x)$ denotes the derivative of $F_j(x)$ with respect to x. Let $F(x)$ be the infinite row vector containing the $F_j(x)$. This system can also be written as

$$F'(x) = F(x)AD^{-1},$$

and its solution is given by

$$F(x) = F(0)e^{AD^{-1}x}.$$

Using a method similar to the uniformization technique, we introduce the *key matrix* G defined by

$$G = I + \frac{AD^{-1}}{\theta}, \quad (4)$$

where

$$\theta = \frac{a+b}{r-c}$$

and I is the identity matrix. We then have, for every $j \geq 0$,

$$F_j(x) = (1 - \rho_0)\sum_{n=0}^{\infty} e^{-\theta x}\frac{(\theta x)^n}{n!}G_{0,j}^n, \quad (5)$$

where $G_{0,j}^n$ denotes the $(0, j)$ entry of matrix G^n. In what follows, we focus on the calculation of $G_{0,j}^n$ using generating functions.

3.1 Generating Functions

Let us consider the complex matrices M indexed on $\mathbb{N} \times \mathbb{N}$. We define

$$\nu(M) = \sup_{i \in \mathbb{N}} \sum_{j=0}^{\infty} |M_{ij}|$$

and denote by $\mathcal{M}$ the set of infinite complex matrices M such that $\nu(M)$ is finite. ν is a norm on $\mathcal{M}$ and $(\mathcal{M}, \nu)$ is a Banach algebra. With each $M \in \mathcal{M}$, we associate the complex function Φ_M, called potential kernel of M or generating function, defined by

$$\Phi_M(z) = \sum_{k=0}^{\infty} M^k z^k$$

for every z such that $|z| < 1/\nu(M)$. Note that for $M \in \mathcal{M}$ and z such that $|z| < 1/\nu(M)$, we have $\Phi_M(z) \in \mathcal{M}$ since

$$\begin{aligned} \nu(\Phi_M(z)) &\leq \sum_{k=0}^{\infty} |z|^k \nu(M^k) \\ &\leq \sum_{k=0}^{\infty} (|z|\nu(M))^k \\ &\leq \frac{1}{1 - |z|\nu(M)} < +\infty. \end{aligned} \quad (6)$$

The following lemma is a classical straightforward result, so we give it without proof.

Lemma 3.1 *For every matrix H, $H\Phi_M$ is the only solution to the matrix equation*

$$X(z) = H + zX(z)M$$

for every z such that $|z| < 1/\nu(M)$.

We shall also need the following result, due to [Leguesdron et al., 1991], which will be used along with Lemma 3.1.

Lemma 3.2 *For every M and N in $\mathcal{M}$, we have*

$$\Phi_{M+N}(z) = \Phi_M(z) + z\Phi_{M+N}(z)N\Phi_M(z)$$

for every z such that

$$|z| < \min\left\{\frac{1}{\nu(M)}, \frac{1}{\nu(M+N)}\right\}.$$

Let us now introduce some notations. We define the infinite matrices V, W, R and S as

$$V_{i,j} = I_{i+1,j} \ , \ W_{i,j} = I_{i,j+1} \ ,$$

$$R_{i,j} = I_{i,0}I_{0,j} \ , \ S_{i,j} = I_{i,0}I_{1,j}$$

for i and $j \in \mathbb{N}$. We studied in [Barbot and Sericola, 2002] the *key matrix* T associated with a fluid buffer fed by an MM1(a,a,b,r) source with service rate c. By comparison with (4), the matrix T is defined by

$$T = I + \frac{1}{\theta}\begin{pmatrix} -a & a & 0 & 0 & \cdots \\ b & -(a+b) & a & 0 & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \end{pmatrix} D^{-1}.$$

Let us define

$$p = \frac{a}{a+b} \ \text{ and } \ q = \frac{b}{a+b}.$$

The non-zero entries of T are then given by

$$T_{0,0} = q + \frac{pr}{c} \ , \ T_{0,1} = p \ ,$$

$$T_{1,0} = q\left(1 - \frac{r}{c}\right) \ , \ T_{1,2} = p \ ,$$

and for $i \geq 2$,

$$T_{i,i-1} = q \ , \ T_{i,i+1} = p \, .$$

We suppose that

$$a \leq \beta \, . \qquad (7)$$

From the assumptions (3) and (7), it follows that the stability condition of the fluid model associated with T is satisfied, that is

$$\rho = \frac{ra}{cb} < 1 \, .$$

Note also that $\rho < 1$ implies that $a < b$ and so $p < q$. After some algebra, we easily obtain the following relation between matrices G and T.

Lemma 3.3 *We have* $G = T + U$ *where*

$$U = (p_0 - p)\left(\frac{r-c}{c}R + S\right)$$

and

$$p_0 = \frac{\beta}{a+b} \, .$$

We can easily check that

$$\begin{aligned} \nu(T) &= \max\left\{1 + p\frac{r}{c}, p + q\frac{r-c}{c}\right\} \\ &< 1 + q\frac{r}{c}, \end{aligned} \qquad (8)$$

since $p < q$. Moreover, from the assumption (7), we have $p \leq p_0$ which implies

$$\nu(U) = (p_0 - p)\frac{r}{c} \, ,$$

and

$$\nu(T) \leq \nu(G) = \nu(T+U) \leq \nu(T) + \nu(U) \, .$$

Using Lemma 3.2, we obtain

$$\Phi_G(z) = \Phi_T(z) + z\Phi_G(z)U\Phi_T(z) \qquad (9)$$

for every z such that $|z| < 1/\nu(G)$. We define the matrix $L(z)$ as

$$L(z) = U\Phi_T(z)$$

for $|z| < 1/\nu(T)$. From (6), we have for $|z| < 1/\nu(T)$

$$\begin{aligned} \nu(L(z)) &= \nu(U\Phi_T(z)) \\ &\leq \frac{\nu(U)}{1 - |z|\nu(T)}, \end{aligned}$$

and so, for every z such as $|z| < 1/(\nu(T) + \nu(U))$,

$$|z| < \frac{1 - |z|\nu(T)}{\nu(U)} \leq \frac{1}{\nu(L(z))}$$

which proves that $L(z) \in \mathcal{M}$. Lemma 3.1 applied to Relation (9) with

$$X(z) = \Phi_G(z) \, , \ H = \Phi_T(z) \ \text{ and } \ M = L(z)$$

leads to

$$\Phi_G(z) = \Phi_T(z)\Phi_{L(z)}(z) \qquad (10)$$

for $|z| < 1/(\nu(T) + \nu(U))$.

In order to derive an expression of the potential kernel Φ_G given in (10), we first recall in the next lemma the expression of Φ_T obtained in [Barbot and Sericola, 2002]. For that, we introduce the generating function $C(z)$ associated with the sequence of the Catalan numbers c_n. For every $n \in \mathbb{N}$, c_n is defined by

$$c_n = \binom{n}{2n}\frac{1}{n+1}$$

and for all z such that $|z| \leq 1/4$, the function $C(z)$ converges and can be written as

$$\begin{aligned} C(z) &= \sum_{n=0}^{\infty} c_n z^n \\ &= \frac{1 - \sqrt{1-4z}}{2z} \, . \end{aligned}$$

Lemma 3.4 *Let $|z| < 1$ and $\eta(z) = C(pqz^2)$. Let $X(z)$ and $Y(z)$ be the matrices defined by*

$$\begin{aligned} X_{i,j}(z) &= (qz\eta(z))^i(pz\eta(z))^j \\ Y(z) &= \sum_{k=0}^{\infty} W^k X(z) V^k . \end{aligned}$$

For every z such that $|z| < \min\{1/2, c/(qr+c)\}$, we have

$$\Phi_T(z) = \eta(z)Y(z) + qz\eta^2(z)\frac{(1+\rho-\rho qz\eta(z))X(z) - \frac{r}{c}WX(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))}. \quad (11)$$

In Lemma 3.4, note that, from (8), the generating function $\Phi_T(z)$ is well-defined for every z such that $|z| < c/(qr+c)$.

Theorem 3.5 *We have for every z such that $|z| < \min\{1/2, c/(qr+c)\}$*

$$L(z) = u(z)RX(z) + \eta(z)(p_0 - p)RX(z)V , \quad (12)$$

$$\Phi_{L(z)}(z) = I + \frac{z}{1 - zu(z)}L(z) , \quad (13)$$

where

$$u(z) = (p_0 - p)\left(\frac{r}{c} - 1\right)\frac{\eta(z)}{1 - \rho qz\eta(z)} .$$

Proof. Let z be such that $|z| < \min\{1/2, c/(qr+c)\}$. We can check, by definition of $X(z)$, that

$$SX(z) = qz\eta(z)RX(z) .$$

Since $RW = 0$ and $SW = R$, we have

$$\begin{aligned} RY(z) &= RX(z) \\ SY(z) &= qz\eta(z)RX(z) + RX(z)V . \end{aligned}$$

These relations give, by definition of matrix U,

$$\begin{aligned} UX(z) &= (p_0 - p)\left(\frac{r}{c} - 1 + qz\eta(z)\right)RX(z) \\ UWX(z) &= (p_0 - p)RX(z) \\ UY(z) &= (p_0 - p)\Big(\left(\frac{r}{c} - 1 + qz\eta(z)\right)RX(z) \\ &\qquad + RX(z)V\Big) . \end{aligned}$$

Lemma 3.4 leads to

$$L(z) = \eta(z)U\left(Y(z) + qz\eta(z)\frac{(1+\rho-\rho qz\eta(z))X(z) - \frac{r}{c}WX(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))}\right)$$

and using the relations above, we obtain (12). Consider now the successive powers $L^k(z)$ of matrix $L(z)$. Observing that $VR = 0$ and

$$X(z)RX(z) = X(z) , \quad (14)$$

we get from (12) that

$$\begin{aligned} L^2(z) &= \Big(u(z)RX(z) + \eta(z)(p_0 - p)RX(z)V\Big)^2 \\ &= u^2(z)RX(z) + u(z)\eta(z)(p_0 - p)RX(z)V \\ &= u(z)L(z) . \end{aligned}$$

It follows by induction that for every $k \geq 0$,

$$L^{k+1}(z) = u^k(z)L(z) .$$

Since $|z| < 1/2$, it is easy to check, from the definition of the function C, that $|\eta(z)| \leq 2$ and therefore $|qz\eta(z)| < 1$. Moreover, since $\rho_0 < 1$, we have

$$(p_0 - p)\left(\frac{r}{c} - 1\right) < q(1 - \rho) \quad (15)$$

and so $|u(z)| < 1$. Thus, we obtain

$$\begin{aligned} \Phi_{L(z)}(z) &= \sum_{k=0}^{\infty}(zL(z))^k \\ &= I + z\sum_{k=0}^{\infty}(zu(z))^k L(z) \\ &= I + \frac{z}{1 - zu(z)}L(z) . \end{aligned}$$

□

Theorem 3.6 *We have for every z such that $|z| < \min\{1/2, c/(qr+c), 1/(\nu(T) + \nu(U))\}$*

$$\begin{aligned} \Phi_G(z) &= \eta(z)Y(z) \\ &+ \eta(z)\frac{qz\eta(z)(1+\rho-\rho qz\eta(z)) + \frac{zu(z)}{1-zu(z)}}{(1-qz\eta(z))(1-\rho qz\eta(z))}X(z) \\ &+ \left(\frac{c}{r-c}\right)\frac{zu(z)}{(1-qz\eta(z))(1-zu(z))}X(z)V \\ &- \left(\frac{r}{c}\right)\frac{qz\eta^2(z)}{(1-qz\eta(z))(1-zu(z))}WX(z) \\ &- \left(\frac{r}{r-c}\right)\frac{qz^2\eta^2(z)u(z)}{(1-qz\eta(z))(1-zu(z))}WX(z)V \quad (16) \end{aligned}$$

Proof. Let z be such that $|z| < \min\{1/2, c/(qr+c), 1/(\nu(T)+\nu(U))\}$.
First, let us observe that, from (12) and (14)

$$\begin{aligned} X(z)L(z) & \\ &= X(z)\Big(u(z)RX(z)+\eta(z)(p_0-p)RX(z)V\Big) \\ &= u(z)X(z)+\eta(z)(p_0-p)X(z)V\,. \qquad (17)\end{aligned}$$

Moreover, since $VR = 0$, we have by definition of $Y(z)$ and relation (14) that

$$Y(z)RX(z) = X(z)\,.$$

It follows from (12) and (17) that

$$\begin{aligned} Y(z)L(z) & \\ &= Y(z)\Big(u(z)RX(z)+\eta(z)(p_0-p)RX(z)V\Big) \\ &= u(z)X(z)+\eta(z)(p_0-p)X(z)V \\ &= X(z)L(z)\,. \qquad (18)\end{aligned}$$

Therefore, from (11), (17) and (18)

$$\begin{aligned} \Phi_T(z)L(z) &= \eta(z)\Bigg(Y(z)+ \\ & qz\eta(z)\frac{(1+\rho-\rho qz\eta(z))X(z)-\frac{r}{c}WX(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))}\Bigg)L(z) \\ &= \eta(z)\frac{X(z)L(z)-\frac{r}{c}WX(z)L(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))} \\ &= \eta(z)\frac{u(z)X(z)+\eta(z)(p_0-p)X(z)V}{(1-qz\eta(z))(1-\rho qz\eta(z))} \\ & -\frac{r}{c}\eta(z)\frac{u(z)WX(z)+\eta(z)(p_0-p)WX(z)V}{(1-qz\eta(z))(1-\rho qz\eta(z))} \qquad (19)\end{aligned}$$

Secondly, from (10) and (13), we have

$$\begin{aligned} \Phi_G(z) &= \Phi_T(z)\Phi_{L(z)}(z) \\ &= \Phi_T(z)\left(I+\frac{z}{1-zu(z)}L(z)\right) \\ &= \Phi_T(z)+\frac{z}{1-zu(z)}\Phi_T(z)L(z)\,. \qquad (20)\end{aligned}$$

Putting (11) and (19) in (20), we obtain the desired result. □

3.2 Explicit Solution For A Single Buffer

We obtain in this section a closed-form expression for $G^n_{0,j}$ and so for

$$\Pr\{Q \le x\} = \sum_{j=0}^{\infty} F_j(x)\,.$$

For that purpose, we need the following well-known lemma which gives an analytical expression of the powers of $\eta(z)$.

Lemma 3.7 *For every $k \ge 1$ and $|z| \le 1/4$, we have*

$$C^k(z) = \sum_{n=0}^{\infty} s(k,n)z^n$$

where $s(k,n)$ are the ballot numbers defined by

$$s(k,n) = k\frac{(2n+k-1)!}{n!(n+k)!}\,.$$

Proof. See e.g. [Riordan, 1968] page 154. □

In order to simplify the writing, let us denote

$$\gamma = (p_0-p)\left(\frac{r}{c}-1\right),$$

which is, from (15), in $[0,1[$. Thus the function $u(z)$ can be written as

$$u(z) = \gamma\frac{\eta(z)}{1-\rho qz\eta(z)}\,. \qquad (21)$$

Theorem 3.8 *For every $x \ge 0$,*

$$\begin{aligned} \Pr\{Q \le x\} &= (1-\rho_0)\sum_{n=0}^{\infty} e^{-\theta x}\frac{(\theta x)^n}{n!} \\ &\times \left(1+(p_0-p)\frac{\theta x}{n+1}\right)\sum_{j=0}^{n}\left(\frac{p}{q}\right)^j\sum_{m=0}^{n-j}\gamma^m \\ &\times \sum_{k=0}^{\lfloor\frac{n-j-m}{2}\rfloor} s(n-2k+1,k)p^kq^{n-m-k} \\ &\times \sum_{h=0}^{n-j-m-2k}\frac{(m+h)!}{h!}\rho^h\end{aligned}$$

where $\lfloor u\rfloor$ denotes the largest integer less than or equal to the real number u.

Proof. Let z be such that $|z| < \min\{1/2, c/(qr+c), 1/(\nu(T)+\nu(U))\}$.
Since the first row of the matrix $WX(z)$ has all its entries equal to zero, we have from Theorem 3.6, for every $j \in \mathbb{N}$,

$$\begin{aligned} (\Phi_G(z))_{0,j} &= \eta(z)Y_{0,j}(z) \\ &+\eta(z)\frac{qz\eta(z)(1+\rho-\rho qz\eta(z))+\dfrac{zu(z)}{1-zu(z)}}{(1-qz\eta(z))(1-\rho qz\eta(z))}X_{0,j}(z)\end{aligned}$$

$$+ \left(\frac{c}{r-c}\right) \frac{zu(z)}{(1-zu(z))(1-qz\eta(z))} (X(z)V)_{0,j}.$$

By definitions of $X(z)$, $Y(z)$ and V, we can easily verify that

$$Y_{0,j}(z) = X_{0,j}(z) = (pz\eta(z))^j,$$

and

$$(X(z)V)_{0,0} = 0, \quad (X(z)V)_{0,j} = (pz\eta(z))^{j-1}.$$

So, we obtain

$$(\Phi_G(z))_{0,0} = \frac{\eta(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))(1-zu(z))} \quad (22)$$

and for $j \geq 1$

$$(\Phi_G(z))_{0,j} = \frac{p^j z^j \eta^{j+1}(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))(1-zu(z))} + \left(\frac{c}{r-c}\right) \frac{p^{j-1} z^j \eta^{j-1}(z) u(z)}{(1-qz\eta(z))(1-zu(z))}. \quad (23)$$

Before inverting the expressions (22) and (23), it must be remembered that for $|x| < 1$ and $n \in \mathbb{N}$

$$(1-x)^{-n-1} = \sum_{l=0}^{\infty} \frac{(n+l)!}{l!} x^l.$$

Since $|z| < 1/2$, we have $|u(z)| < 1$ and $|qz\eta(z)| < 1$. Therefore, using the Cauchy product of two series, we obtain from (22)

$$\begin{aligned}
(\Phi_G(z))_{0,0} &= \frac{\eta(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))} \sum_{n=0}^{\infty} (zu(z))^n \\
&= \frac{\eta(z)}{1-qz\eta(z)} \sum_{n=0}^{\infty} (\gamma z\eta(z))^n (1-\rho qz\eta(z))^{-n-1} \\
&= \frac{\eta(z)}{1-qz\eta(z)} \sum_{n,l=0}^{\infty} (\gamma z\eta(z))^n \frac{(n+l)!}{l!} (\rho qz\eta(z))^l \\
&= \eta(z) \sum_{n,l=0}^{\infty} (\gamma z\eta(z))^n \sum_{h=0}^{l} \frac{(n+h)!}{h!} (\rho qz\eta(z))^h \\
&\quad \times (qz\eta(z))^{l-h} \\
&= \sum_{n,l=0}^{\infty} z^{n+l} \eta^{n+l+1}(z) \gamma^n q^l \sum_{h=0}^{l} \frac{(n+h)!}{h!} \rho^h. \quad (24)
\end{aligned}$$

From Lemma 3.7, we have for every $m \in \mathbb{N}$

$$\begin{aligned}
\eta^{m+1}(z) &= C^{m+1}(pqz^2) \\
&= \sum_{k=0}^{\infty} s(m+1,k) p^k q^k z^{2k} \quad (25)
\end{aligned}$$

which leads, by changing the order of summations, to

$$\begin{aligned}
(\Phi_G(z))_{0,0} &= \sum_{n,l,k=0}^{\infty} z^{n+l+2k} s(n+l+1,k) \gamma^n p^k q^{l+k} \\
&\quad \times \sum_{h=0}^{l} \frac{(n+h)!}{h!} \rho^h \\
&= \sum_{n=0}^{\infty} z^n \sum_{m=0}^{n} \gamma^m \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} p^k q^{n-m-k} \\
&\quad \times s(n-2k+1,k) \sum_{h=0}^{n-m-2k} \frac{(m+h)!}{h!} \rho^h.
\end{aligned}$$

Then, we have for every $n \in \mathbb{N}$,

$$G_{0,0}^n = \sum_{m=0}^{n} \gamma^m \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} p^k q^{n-m-k} \times s(n-2k+1,k) \sum_{h=0}^{n-m-2k} \frac{(m+h)!}{h!} \rho^h. \quad (26)$$

Similary, for $j \geq 1$, we have from (21), (22) and (23)

$$\begin{aligned}
(\Phi_G(z))_{0,j} &= \frac{p^j z^j \eta^{j+1}(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))(1-zu(z))} \\
&+ \left(\frac{c}{r-c}\right) \gamma \frac{p^{j-1} z^j \eta^j(z)}{(1-qz\eta(z))(1-\rho qz\eta(z))(1-zu(z))} \\
&= \left(p^j z^j \eta^j(z) + (p_0 - p) p^{j-1} z^j \eta^{j-1}(z)\right) (\Phi_G(z))_{0,0}
\end{aligned}$$

which gives, using (24)

$$\begin{aligned}
(\Phi_G(z))_{0,j} &= \\
& z^j p^j \sum_{n,l=0}^{\infty} z^{n+l} \eta^{n+l+j+1}(z) \gamma^n q^l \sum_{h=0}^{l} \frac{(n+h)!}{h!} \rho^h + \\
& z^j p^{j-1} (p_0 - p) \sum_{n,l=0}^{\infty} z^{n+l} \eta^{n+l+j}(z) \gamma^n q^l \sum_{h=0}^{l} \frac{(n+h)!}{h!} \rho^h.
\end{aligned}$$

From (25) and changing the order of summations, we get

$$\begin{aligned}
(\Phi_G(z))_{0,j} &= z^j p^j \sum_{n=0}^{\infty} z^n \sum_{m=0}^{n} \gamma^m \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} p^k q^{n-m-k} \\
&\times \; s(n-2k+j+1,k) \sum_{h=0}^{n-m-2k} \frac{(m+h)!}{h!} \rho^h \\
&+ \; z^j p^{j-1} (p_0 - p) \sum_{n=0}^{\infty} z^n \sum_{m=0}^{n} \gamma^m \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} p^k q^{n-m-k} \\
&\times \; s(n-2k+j,k) \sum_{h=0}^{n-m-2k} \frac{(m+h)!}{h!} \rho^h.
\end{aligned}$$

Therefore, $G_{0,j}^{n} = 0$ if $n < j$, and for $n \geq j$

$$\begin{aligned} G_{0,j}^{n} &= \left(\frac{p}{q}\right)^{j} \sum_{m=0}^{n-j} \gamma^{m} \sum_{k=0}^{\lfloor \frac{n-j-m}{2} \rfloor} p^{k} q^{n-m-k} \\ &\times \quad s(n-2k+1,k) \sum_{h=0}^{n-j-m-2k} \frac{(m+h)!}{h!} \rho^{h} \\ &+ \quad \frac{p_0 - p}{q} \left(\frac{p}{q}\right)^{j-1} \sum_{m=0}^{n-j} \gamma^{m} \sum_{k=0}^{\lfloor \frac{n-j-m}{2} \rfloor} p^{k} q^{n-m-k} \\ &\times \quad s(n-2k,k) \sum_{h=0}^{n-j-m-2k} \frac{(m+h)!}{h!} \rho^{h} . \qquad (27) \end{aligned}$$

Putting Relations (26) and (27) in (5), we have by summing over j

$$\begin{aligned} \Pr\{Q \leq x\} &= (1-\rho_0) \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} e^{-\theta x} \frac{(\theta x)^{n}}{n!} \\ &\times \left(\frac{p}{q}\right)^{j} \sum_{m=0}^{n-j} \gamma^{m} \sum_{k=0}^{\lfloor \frac{n-j-m}{2} \rfloor} p^{k} q^{n-m-k} \\ &\times s(n-2k+1,k) \sum_{h=0}^{n-j-m-2k} \frac{(m+h)!}{h!} \rho^{h} \\ &+ (1-\rho_0) \frac{p_0 - p}{q} \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} e^{-\theta x} \frac{(\theta x)^{n}}{n!} \\ &\times \left(\frac{p}{q}\right)^{j-1} \sum_{m=0}^{n-j} \gamma^{m} \sum_{k=0}^{\lfloor \frac{n-j-m}{2} \rfloor} p^{k} q^{n-m-k} \\ &\times s(n-2k,k) \sum_{h=0}^{n-j-m-2k} \frac{(m+h)!}{h!} \rho^{h} \\ &= (1-\rho_0) \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} e^{-\theta x} \frac{(\theta x)^{n}}{n!} \\ &\times \left(\frac{p}{q}\right)^{j} \sum_{m=0}^{n-j} \gamma^{m} \sum_{k=0}^{\lfloor \frac{n-j-m}{2} \rfloor} p^{k} q^{n-m-k} \\ &\times s(n-2k+1,k) \sum_{h=0}^{n-j-m-2k} \frac{(m+h)!}{h!} \rho^{h} \\ &+ (1-\rho_0) \frac{p_0 - p}{q} \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} e^{-\theta x} \frac{(\theta x)^{n+1}}{(n+1)!} \\ &\times \left(\frac{p}{q}\right)^{j} \sum_{m=0}^{n-j} \gamma^{m} \sum_{k=0}^{\lfloor \frac{n-j-m}{2} \rfloor} p^{k} q^{n-m-k+1} \\ &\times s(n-2k,k) \sum_{h=0}^{n-j-m-2k} \frac{(m+h)!}{h!} \rho^{h} . \end{aligned}$$

Changing the order of summations, we obtain the result. □

4 THE FLUID CONTENT OF THE $(i+1)$TH BUFFER

We suppose that the stability condition of the tandem fluid queues is satisfied, that is, $\rho_M < 1$. For $i = 1, \ldots, M-1$, we derive the stationary distribution of the fluid level Q_{i+1} of the $(i+1)$th buffer.

Theorem 4.1 *For every $x \geq 0$ and $1 \leq i \leq M-1$*

$$\begin{aligned} \Pr\{Q_{i+1} \leq x\} &= (1-\rho_{i+1}) \sum_{n=0}^{\infty} e^{-\theta_i x} \frac{(\theta_i x)^{n}}{n!} \\ &\times \left(1 + \frac{\lambda x}{(n+1)c_0(1 - c_{i+1}/c_i)}\right) \sum_{j=0}^{n} \left(\frac{p_i}{q_i}\right)^{j} \\ &\times \sum_{m=0}^{n-j} \gamma_i^{m} \sum_{k=0}^{\lfloor \frac{n-j-m}{2} \rfloor} s(n-2k+1,k) p_i^{k} q_i^{n-m-k} \\ &\times \sum_{h=0}^{n-j-m-2k} \frac{(m+h)!}{h!} \rho_i'^{h} \end{aligned}$$

where

$$\begin{aligned} p_i &= \frac{\lambda_i}{\lambda_i + \mu_i} \\ q_i &= 1 - p_i \\ \theta_i &= \frac{\lambda_i + \mu_i}{c_i - c_{i+1}} \\ \rho_i' &= \frac{c_i \lambda_i}{c_{i+1} \mu_i} \\ \gamma_i &= \frac{c_i (c_i - c_{i+1}) \lambda}{c_0 c_{i+1} (\lambda_i + \mu_i)} . \end{aligned}$$

Proof. We saw that the stationary level of the $(i+1)$th buffer is equivalent to the stationary level of an infinite buffer with service rate c_{i+1} and fed by an MM1($N\lambda$,λ_i,μ_i,c_i) source. We can then apply Theorem 3.8 to this fluid model by setting $\beta = N\lambda$, $a = \lambda_i$, $b = \mu_i$, $r = c_i$ and $c = c_{i+1}$, since the assumption (7) is satisfied. □

This solution can be generalized to the case of tandem fluid queues driven by an M/M/∞ queue: let $N \to \infty$ and $\lambda \to 0$ in such a way that $N\lambda$ converges to a finite limit Λ. The stationary distribution of the first buffer has been derived in [Kosten, 1974], via spectral decomposition arguments. It has been proven in [Aalto, 1998] that the output process of each buffer can be modeled as an on-off source modulated by an M/M/1 queue. It is immediate to check that the solution in Theorem 4.1 for the stationary amount of fluid in the $(i+1)$th buffer converges to the same one derived in [Barbot and Sericola, 2002] for a fluid buffer driven by an M/M/1 queue (more precisely by an MM1(Λ,Λ,μ_i,c_i) source), that is

$$\Pr\{Q_{i+1} \leq x\} = \sum_{n=0}^{\infty} e^{-\theta_i x} \frac{(\theta_i x)^n}{n!} \sum_{j=0}^{n} \left(\frac{p_i}{q_i}\right)^j \times \sum_{k=0}^{\lfloor \frac{n-j}{2} \rfloor} s(n-2k+1,k) p_i^k q_i^{n-k} \left(1 - \rho_{i+1}^{n-j-2k+1}\right)$$

where

$$\begin{aligned} p_i &= \frac{\Lambda}{\Lambda + \mu_i} \\ q_i &= 1 - p_i \\ \theta_i &= \frac{\Lambda + \mu_i}{c_i - c_{i+1}} \\ \rho_{i+1} &= \frac{c_i \Lambda}{c_{i+1} \mu_i}. \end{aligned}$$

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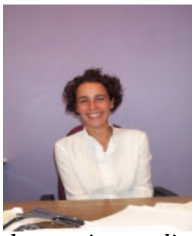

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