PRODUCT FORM SOLUTION FOR A G-NETWORK WITH SIGNALS AND IMPATIENT SERVICE

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Abstract: This paper presents some results on an exponential queuing network with signals and impatient service. Positive customers and signals arrive to each node according to a Poisson process. When the service is finished in a node, a positive customer moves to another node with fixed probabilities either as a positive customer or as a signal, or quits the network. Every signal is activated during a random exponentially distributed amount of time. Activated signals with fixed probabilities either move a customer from the node they arrive to another node or kill a positive customer. Each customer can be served in a node at most a random time ("patient" time) distributed exponentially. When the patient service is finished, the customer with fixed probabilities either goes to another node or quits the network. Product form solution has been obtained for stationary state probabilities of such G-network in the case of positive customers processed by a single server in each node as well as in the case of an analogous symmetrical G-network in which service rate of a positive customer in a node depends on its state.

Key Words: G-networks, positive customers, impatient service, product form solution

1 Introduction

G-networks models have been introduced by Gelenbe [Gelenbe, 1991; Gelenbe, 1993], and they have been motivated by the analogy with neural networks. Among a wide range of applications they have been applied to evaluate the performance of unreliable flow systems and to describe virus behaviour in a computer network. A G-network is characterized by positive and negative customers, signals and triggers. Positive customers are ordinary customers which join a queue in order to receive service and which can be destroyed by a negative customer arriving to the queue. Negative customers joining a non-empty queue have the effect to destroy a positive customer. The role of a trigger is to displace a positive customer from a queue to another. A signal combines these two kinds of customers and can act either as a negative customer or a trigger. In [Artalejo 2000] and [Bocharov-Vishnevskii 2002] a vast review of works on G-networks is done.

In this paper we deal with a queuing network with positive customers and signals. Positive customers and signals arrive from the outside at each node according to two independent Poisson processes. Service times of positive customers at each node are exponentially distributed. After receiving service, a positive customer goes from a node to another node with fixed probabilities either as a positive customer or as a signal or quits the network. The activation time of a signal is exponentially distributed. Activated signals with fixed probabilities either move a customer from the node they arrive to another node or kill a positive customer.

A similar G-network with instantaneous signal activation is analyzed in [Gelenbe and Pujolle, 1998]. An analogous system with random signal activation period was studied in [Bocharov 2002]. In [Gelenbe and Pujolle, 1998] and [Bocharov 2002] formulas in product-form have been derived for the stationary state distribution. We introduce in the described model a "patient" time. Each customer can be served in a node at most a random time ("patient" time) exponentially distributed. When the patient time is finished the customer goes to another node or leaves the network with fixed probabilities. Stationary state distribution is derived in product form in the case of positive customers processed by a single server and in the case of a symmetrical network in which service rate of a positive customer at each node depends on the number of positive customers in this node.

2 Model description

We consider G-networks with a finite number of nodes M, positive customers and signals. External arrival flows to the network are independent Poisson processes. We denote, respectively, with λ_{0i}^{+} and λ_{0i}^{-} the arrival rate of external positive customers and external signals at node i.

The service of a positive customer is finished at node i with probability $\mu_i^{+}(k)\Delta + o(\Delta)$ in an interval time $(t, t+\Delta)$, provided that there are k positive customers at this node at instant t. A positive customers who leaves queue i moves from node i to node j with probability p_{ij}^{+} as a positive customer, and with probability p_{ij}^{-} as a signal. He quits the network with probability $p_{i0} = 1 - \sum_{j=1}^{M}(p_{ij}^{+} + p_{ij}^{-})$.

Every signal is activated during a random time. A signal arriving at node i is activated in a time interval $(t, t+\Delta)$ with probability $\mu_i^{-}(n)\Delta + o(\Delta)$, provided that n non activated signals are present at this node at instant t. After finishing activation period a signal:

- with probability q_{ij}^{+} acts as a trigger, i.e. he routes a positive customer from node i to node j retaining him as a positive customer;
- with probability q_{ij}^{-} moves a positive customer from node i to node j retaining him as a signal;
- with probability q_{i0} he acts as a negative customers, i.e. he kills a positive customer at node i and he vanishes.

An activated signal in node i who finds no positive customers disappears without any other effects. The impatience time of a customer in the node i is completed in a time interval $(t, t+\Delta)$ with probability $\gamma_i(k)\Delta + o(\Delta)$, provided that k positive customers are present at this node at instant t. Then a positive customer with probability r_{ij}^{+} goes from node i to node j as a positive customer, with probability r_{ij}^{-} as a signal, and with probability $r_{i0} = 1 - \sum_{j=1}^{M}(r_{ij}^{+} + r_{ij}^{-})$ he leaves the network.

3 Equilibrium equations

Let us denote with P^{+}, P^{-}, Q^{+}, Q^{-}, R^{+}, R^{-} the matrices with elements p_{ij}^{+}, p_{ij}^{-}, q_{ij}^{+}, q_{ij}^{-}, r_{ij}^{+}, r_{ij}^{-}, respectively, $i, j = \overline{1, M}$, and let us set $P = P^{+} + P^{-}$, $Q = Q^{+} + Q^{-}$, and $R = R^{+} + R^{-}$. The stochastic behaviour of the queueing network under consideration can be described by an homogeneous Markov process $\{X(t),\ t \geq 0\}$ with the following state space:

$$\mathcal{X} = \{((k_1, n_1), ..., (k_M, n_M)), k_i \geq 0, n_i \geq 0, i = \overline{1, M}\}. \quad (1)$$

The state $((k_1, n_1), (k_2, n_2), ..., (k_M, n_M))$ means that at any instant there are k_1 positive customers and n_1 non-activated signals at node 1, k_2 customers and n_2 signals at node 2, ..., and finally, k_M customers and n_M signals at node M.

Let us introduce $(\vec{k}, \vec{n}) = ((k_1, n_1), ..., (k_M, n_M))$ where $\vec{k} = (k_1, k_2, ..., k_M)$ and $\vec{n} = (n_1, n_2, ..., n_M)$ and $\vec{e}_i$ the vector with $i-th$ component equal to 1 and other components equals to 0. We also introduce the notation $\lambda_0^{+} = \sum_{i=1}^{M}\lambda_{0i}^{+}$ and $\lambda_0^{-} = \sum_{i=1}^{M}\lambda_{0i}^{-}$.

Let $p(\vec{k}, \vec{n})$ denote the stationary probability of the state $(\vec{k}, \vec{n})$. If the stationary distribution $\{p(\vec{k}, \vec{n}),\ \ \vec{k}, \vec{n} \geq \vec{0}\}$ of the process $\{X(t),\ \ t \geq 0\}$ exists then the following equilibrium equations system holds:

$$\begin{aligned}
&p(\vec{k}, \vec{n})(\lambda_0^{+} + \lambda_0^{-} + \sum_{i=1}^{M}\mu_i^{+}(k_i)(1 - p_{ii}^{+}) + \sum_{i=1}^{M}\mu_i^{-}(n_i) \\
&+ \sum_{i=1}^{M}\gamma_i(k_i)(1 - \gamma_{ii}^{+})) = \sum_{i=1}^{M}p(\vec{k} - \vec{e}_i, \vec{n})\lambda_{0i}^{+}u(k_i) + \\
&\sum_{i=1}^{M}p(\vec{k}, \vec{n} - \vec{e}_i)\lambda_{0i}^{-}u(n_i) + \\
&\sum_{i=1}^{M}p(\vec{k} + \vec{e}_i, \vec{n})\mu_i^{+}(k_i + 1)p_{i0} + \\
&\sum_{i=1}^{M}p(\vec{k} + \vec{e}_i, \vec{n} + \vec{e}_i)\mu_i^{-}(n_i + 1)q_{i0} + \\
&\sum_{i=1}^{M}p(\vec{k} + \vec{e}_i, \vec{n})\gamma_i(k_i + 1)r_{i0} + \\
&\sum_{i=1}^{M}p(\vec{k}, \vec{n} + \vec{e}_i)\mu_i^{-}(n_i + 1)(1 - u(k_i)) + \\
&\sum_{i=1}^{M}\sum_{j=1, j\neq i}^{M}p(\vec{k} + \vec{e}_i - \vec{e}_j, \vec{n})\mu_i^{+}(k_i + 1)p_{ij}^{+}u(k_j) + \\
&\sum_{i=1}^{M}\sum_{j=1}^{M}p(\vec{k} + \vec{e}_i, \vec{n} - \vec{e}_j)\mu_i^{+}(k_i + 1)p_{ij}^{-}u(n_j) + \\
&\sum_{i=1}^{M}\sum_{j=1}^{M}p(\vec{k} + \vec{e}_i - \vec{e}_j, \vec{n} + \vec{e}_i)\mu_i^{-}(n_i + 1)q_{ij}^{+}u(k_j) + \\
&\sum_{i=1}^{M}\sum_{j=1, j\neq i}^{M}p(\vec{k} + \vec{e}_i, \vec{n} + \vec{e}_i - \vec{e}_j)\mu_i^{-}(n_i + 1)q_{ij}^{-}u(n_j) +
\end{aligned} \quad (2)$$

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$$\sum_{i=1}^{M} p(\vec{k}+\vec{e}_i,\vec{n})\mu_i^-(n_i)q_{ii}^- +$$
$$\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M} p(\vec{k}+\vec{e}_i-\vec{e}_j,\vec{n})\gamma_i(k_i+1)r_{ij}^+u(k_j)+$$
$$\sum_{i=1}^{M}\sum_{j=1}^{M} p(\vec{k}+\vec{e}_i,\vec{n}-\vec{e}_j)\gamma_i(k_i+1)r_{ij}^-u(n_j),$$
$$(\vec{k},\vec{n})\in\mathcal{X}, \quad (3)$$

where $\mu_i^+(0)=0$, $\mu_i^-(0)=0$, $\gamma_i(0)=0$ and $u(x)$ is a unit Heavyside function.

4 Single server node case and symmetrical case

We are not able to find the general product-form of the system of equations (3). Nevertheless, such solutions for two important cases have been derived.

4.1 Service of positive customers by a single server

Consider a network in which positive customers are served at every node by a single server and the service time at node i is exponentially distributed with parameter μ_i^+. Therefore

$$\mu_i^+(k_i)=u(k_i)\mu_i^+, \quad i=\overline{1,M}. \quad (4)$$

We also assume that

$$\gamma_i(k_i)=u(k_i)\gamma_i, \quad i=\overline{1,M}. \quad (5)$$

Let us introduce the following notations:

$$q_i=\frac{\lambda_i^+}{\lambda_i^-+\mu_i^++\gamma_i}, \quad \rho_i^-(j)=\frac{\lambda_i^-}{\mu_i^-(j)}, \quad i,j=\overline{1,M}. \quad (6)$$

$$\lambda_i^+=\lambda_{0i}^+ + \sum_{j=1}^{M} q_j(\mu_j^+p_{ji}^+ + \lambda_j^-q_{ji}^+ + \gamma_jr_{ji}^+),\ i=\overline{1,M},$$
$$\lambda_i^-=\lambda_{0i}^- + \sum_{j=1}^{M} q_j(\mu_j^+p_{ji}^- + \lambda_j^-q_{ji}^- + \gamma_jr_{ji}^-),\ i=\overline{1,M}. \quad (7)$$

As in [Gelenbe and Pujolle, 1998] we can prove that there exists a unique positive solution λ_i^+, λ_i^-, $i=\overline{1,M}$, of the system of equations (7). Besides let us denote

$$\Lambda_0=\sum_{j=1}^{M} q_j\mu_j^+p_{j0} + \sum_{j=1}^{M} q_j\lambda_j^-q_{j0} + \sum_{j=1}^{M} q_j\gamma_jr_{j0}. \quad (8)$$

From (6) - (8) we obtain

$$\Lambda_0+\sum_{j=1}^{M}(\lambda_j^++\lambda_j^-)=\lambda_0^++\lambda_0^-+\sum_{j=1}^{M} q_j(\mu_j^++\lambda_j^-+\gamma_j)$$
$$=\lambda_0^++\lambda_0^-+\sum_{j=1}^{M}\lambda_j^+. \quad (9)$$

Therefore

$$\Lambda_0+\sum_{j=1}^{M}\lambda_j^-=\lambda_0^++\lambda_0^-. \quad (10)$$

The following theorem holds.

Theorem 1

If the matrices P, Q, and R are irreducible, conditions (4) and (5) hold, and a unique positive solution of equations (7) exists such that

$$\lambda_i^+<\lambda_i^-+\mu_i^++\gamma_i, \quad i=\overline{1,M}, \quad (11)$$

$$G_i=\sum_{n_i=0}^{\infty}\prod_{j=1}^{n_i}\rho_i^-(j)<\infty, \quad i=\overline{1,M}, \quad (12)$$

then the Markov process $\{X(t), \quad t\geq 0\}$ is ergodic and its stationary distribution is represented in a product form as

$$p(\vec{k},\vec{n})=\prod_{i=1}^{M}p(k_i,n_i), \quad (13)$$

where

$$p(k_i,n_i)=(1-q_i)q_i^{k_i}G_i^{-1}\prod_{j=1}^{n_i}\rho_i^-(j), \quad k_i,n_i\geq 0. \quad (14)$$

and $\prod_{j=1}^{0}\equiv 1$.

Proof

The substitution of expressions (13), (14), (6) for the stationary distribution of the process $\{X(t), \quad t\geq 0\}$ into the equilibrium system of equations (3) leads to the following equalities:

$$\lambda_0^+ + \lambda_0^- + \sum_{i=1}^M \mu_i^+ u(k_i) + \sum_{i=1}^M \mu_i^-(n_i) + \sum_{i=1}^M \gamma_i u(k_i) =$$
$$\sum_{i=1}^M \frac{\lambda_{0i}^+}{q_i} u(k_i) + \sum_{i=1}^M \frac{\mu_i^-(n_i)}{\lambda_i^-}\lambda_{0i}^- + \sum_{i=1}^M q_i\mu_i^+ p_{i0} +$$
$$\sum_{i=1}^M q_i\lambda_i^- q_{i0} + \sum_{i=1}^M q_i\gamma_i r_{i0} + \sum_{i=1}^M \lambda_i^-(1-u(k_i)) +$$
$$\sum_{i=1}^M\sum_{j=1}^M \frac{q_i}{q_j}\mu_i^+ p_{ij}^+ u(k_j) +$$
$$\sum_{i=1}^M\sum_{j=1}^M q_i \frac{\mu_j^-(n_j)}{\lambda_j^-}\mu_i^+ p_{ij}^- + \sum_{i=1}^M\sum_{j=1}^M \frac{q_i}{q_j}\lambda_i^- q_{ij}^+ u(k_j) +$$
$$\sum_{i=1}^M\sum_{j=1}^M q_i \frac{\mu_j^-(n_j)}{\lambda_j^-}\lambda_i^- q_{ij}^- + \sum_{i=1}^M\sum_{j=1}^M \frac{q_i}{q_j}\gamma_i r_{ij}^+ u(k_j) +$$
$$\sum_{i=1}^M\sum_{j=1}^M q_i \frac{\mu_j^-(n_j)}{\lambda_j^-}\gamma_i r_{ij}^-. \tag{15}$$

The latter equalities takes place for all $(\vec{k},\vec{n}) \in \mathcal{X}$. Let us denote by

$$A = \sum_{i=1}^M \frac{\mu_i^-(n_i)}{\lambda_i^-}\lambda_{0i}^- + \sum_{i=1}^M\sum_{j=1}^M q_i \frac{\mu_j^-(n_j)}{\lambda_j^-}\mu_i^+ p_{ij}^- +$$
$$\sum_{i=1}^M\sum_{j=1}^M q_i \frac{\mu_j^-(n_j)}{\lambda_j^-}\lambda_i^- q_{ij}^- + \sum_{i=1}^M\sum_{j=1}^M q_i \frac{\mu_j^-(n_j)}{\lambda_j^-}\gamma_i r_{ij}^-.$$

Taking into account (7) we obtain

$$A = \sum_{j=1}^M \mu_j^-(n_j). \tag{16}$$

Further let us denote by

$$B = \sum_{i=1}^M \frac{\lambda_{0i}^+}{q_i} u(k_i) + \sum_{i=1}^M\sum_{j=1}^M \frac{q_i}{q_j}\mu_i^+ p_{ij}^+ u(k_j) +$$
$$\sum_{i=1}^M\sum_{j=1}^M \frac{q_i}{q_j}\lambda_i^- q_{ij}^+ u(k_j) + \sum_{i=1}^M\sum_{j=1}^M \frac{q_i}{q_j}\gamma_i r_{ij}^+ u(k_j).$$

After some transformations of the right part with combination with (7) we obtain

$$B = \sum_{j=1}^M \frac{\lambda_j^- + \mu_j^+ + \gamma_j}{\lambda_j^+}[\lambda_{0j}^+ +$$
$$\sum_{i=1}^M q_i(\mu_i^+ p_{ij}^+ + \lambda_i^- q_{ij}^+ + \gamma_i r_{ij}^+)]u(k_j) = \tag{17}$$
$$\sum_{j=1}^M (\lambda_j^- + \mu_j^+ + \gamma_j)u(k_j).$$

Finally let us introduce

$$C = \Lambda_0 + \sum_{i=1}^M \lambda_i^-(1-u(k_i)). \tag{18}$$

The right part of equalities (15) can be represented as $A+B+C$. Then we have

$$A+B+C =$$
$$\sum_{j=1}^M \mu_j^-(n_j) + \sum_{j=1}^M (\lambda_j^- + \mu_j^+ + \gamma_j)u(k_j) + \Lambda_0 +$$
$$\sum_{i=1}^M \lambda_i^- - \sum_{i=1}^M \lambda_i^- u(k_i) =$$
$$\sum_{j=1}^M \mu_j^-(n_j) + \sum_{j=1}^M (\mu_j^+ + \gamma_j)u(k_j) + \Lambda_0 + \sum_{i=1}^M \lambda_i^- =$$
$$\lambda_0^+ + \lambda_0^- + \sum_{j=1}^M \mu_j^-(n_j) + \sum_{j=1}^M (\mu_j^+ + \gamma_j)u(k_j).$$

This coincides with the left part of equalities (15). Thus the substitution of (13), (14) into the system of equations (3) - (5) leads to a system of identities for all $(\vec{k},\vec{n}) \in \mathcal{X}$. Under the assumptions of the theorem the expressions (13), (14) determine a positive solution of the equilibrium system of equations (3) - (5) and this solution is bounded. Moreover under theorem assumptions the process $\{X(t), \quad t \geq 0\}$ is irreducible. Therefore, according to Foster's theorem the process is ergodic and the relations (13), (14) give us its unique stationary distribution. Thus, the theorem is proved.

4.2 Symmetrical network

We consider the network described in the section 2 with

$$p_{ij}^+ = q_{ij}^+ = r_{ij}^+,\ p_{ij}^- = q_{ij}^- = r_{ij}^-,\ p_{i0} = q_{i0} = r_{i0}, \tag{19}$$
$$i,j = \overline{1,M}.$$

It is convenient to call a queueing network under these conditions as a symmetrical network. Let us introduce the following notations:

$$q_i(j) = \frac{\lambda_i^+}{\lambda_i^- + \mu_i^+(j) + \gamma_i(j)}, \tag{20}$$
$$\rho^-(j) = \frac{\lambda_i^-}{\mu_i^-(j)}, \quad i,j = \overline{1,M},$$

$$\lambda_i^+ = \lambda_{0i}^+ + \sum_{j=1}^M \lambda_j^+ p_{ji}^+, \quad i = \overline{1,M},$$
$$\lambda_i^- = \lambda_{0i}^- + \sum_{j=1}^M \lambda_j^+ p_{ji}^-, \quad i = \overline{1,M}. \tag{21}$$

If the matrix P is irreducible, the system (21) has a unique positive solution for λ_i^+, λ_i^-, $i = \overline{1,M}$.
Let us denote

$$\Lambda_0 = \sum_{i=1}^M \lambda_i^+ p_{i0}. \tag{22}$$

From (21) and (22) we obtain

$$\Lambda_0 + \sum_{j=1}^{M}(\lambda_j^+ + \lambda_j^-) = \lambda_0^+ + \lambda_0^- + \sum_{j=1}^{M}\lambda_j^+. \qquad (23)$$

This yields

$$\Lambda_0 + \sum_{j=1}^{M}\lambda_j^+ = \lambda_0^+ + \lambda_0^-. \qquad (24)$$

The relation (24) formally coincides with relation (10) obtained for the case of single-server processing of positive customers but the values of λ_i^+ and λ_i^- for the symmetrical network are determined from another system of equations which is a linear one.

Theorem 2

it If matrix P is irreducible and the following conditions hold :

$$F_i = \sum_{k_i=0}^{\infty}\prod_{j=1}^{k_i} q_i(j) < \infty, \quad G_i = \sum_{n_i=0}^{\infty}\prod_{j=1}^{n_i}\rho^-(j) < \infty,$$
$$i = \overline{1,M}.$$

then the Markov process $\{X(t), \quad t \geq 0\}$ *is ergodic and its stationary distribution is represented in a product form as*

$$p(\vec{k},\vec{n}) = \prod_{i=1}^{M} p(k_i,n_i), \qquad (25)$$

where

$$p(k_i,n_i) = F_i^{-1}G_i^{-1}\prod_{j=1}^{k_i} q_i(j)\prod_{l=1}^{n_i}\rho_l^-(j), \quad k_i,n_i \geq 0. \qquad (26)$$

Proof

We make the substitution of (25), (26) into the system of equations (3), for which the assumptions (19) take place.

After some algebraic transformations we obtain the equality

$$\lambda_0^+ + \lambda_0^- + \sum_{i=1}^{M}\mu_i^+(k_i) + \sum_{i=1}^{M}\mu_i^-(n_i) + \sum_{i=1}^{M}\gamma_i(k_i) =$$
$$\sum_{i=1}^{M}\mu_i^+(k_i)p_{ii}^+ + \sum_{i=1}^{M}\gamma_i(k_i)p_{ii}^+ + \sum_{i=1}^{M}\frac{\lambda_{0i}^+}{q_i(k_i)}u(k_i)+$$
$$\sum_{i=1}^{M}\frac{\mu_i^-(n_i)}{\lambda_i^-}\lambda_{0i}^-+$$
$$\sum_{i=1}^{M} q_i(k_i+1)\mu_i^+(k_i+1)p_{i0} + \sum_{i=1}^{M} q_i(k_i+1)\lambda_i^- p_{i0}+$$
$$\sum_{i=1}^{M} q_i(k_i+1)\gamma_i^+(k_i+1)p_{i0} + \sum_{i=1}^{M}\lambda_i^-(1-u(k_i))+$$
$$\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\frac{q_i(k_i+1)}{q_j(k_j)}\mu_i^+(k_i+1)p_{ij}^+u(k_j)+$$
$$\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\mu_j^-(n_j)}{\lambda_j^-}q_i(k_i+1)\mu_i^+(k_i+1)p_{ij}^-+$$
$$\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\frac{q_i(k_i+1)}{q_j(k_j)}\lambda_i^- p_{ij}^+u(k_j)+$$
$$\sum_{i=1}^{M}\lambda_i^- p_{ii}^+u(k_i) + \sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\mu_j^-(n_j)}{\lambda_j^-}q_i(k_i+1)\lambda_i^- p_{ij}^-+$$
$$\sum_{i=1}^{M}\sum_{j=1,j\neq i}^{M}\frac{q_i(k_i+1)}{q_j(k_j)}\gamma_i(k_i+1)p_{ij}^+u(k_j)+$$
$$\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\mu_j^-(n_j)}{\lambda_j^-}q_i(k_i+1)\gamma_i(k_i+1)p_{ij}^-. \qquad (27)$$

This equality is true for all $(\vec{k},\vec{n}) \in \mathcal{X}$.

Similarly to the proof of the theorem of the previous case we transform the right part of the equality (27). Let us denote by

$$A = \sum_{i=1}^{M}\frac{\mu_i^-(n_i)}{\lambda_i^-}\lambda_{0i}^-+$$
$$\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\mu_j^-(n_j)}{\lambda_j^-}q_i(k_i+1)\mu_i^+(k_i+1)p_{ij}^-+$$
$$\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\mu_j^-(n_j)}{\lambda_j^-}q_i(k_i+1)\lambda_i^- p_{ij}^-+$$
$$\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\mu_j^-(n_j)}{\lambda_j^-}q_i(k_i+1)\gamma_i(k_i+1)p_{ij}^-.$$

Taking into account the relation

$$q_i(k_i+1)[\lambda_i^- + \mu_i^+(k_i+1) + \gamma_i(k_i+1)] = \lambda_i^+,$$

we obtain

$$A = \sum_{j=1}^{M}\mu_j^-(n_j). \qquad (28)$$

Further let us denote by

$$B = \sum_{i=1}^{M} \mu_i^+(k_i)p_{ii}^+ + \sum_{i=1}^{M} \frac{\lambda_{0i}^+}{q_i(k_i)} u(k_i) + \sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \frac{q_i(k_i+1)}{q_j(k_j)} \mu_i^+(k_i+1) p_{ij}^+ u(k_j) + \sum_{i=1}^{M} \sum_{j=1, j\neq i}^{M} \frac{q_i(k_i+1)}{q_j(k_j)} \lambda_i^- p_{ij}^+ u(k_j) + \sum_{i=1}^{M} \lambda_i^- p_{ii}^+ u(k_i) + \sum_{i=1}^{M} \gamma_i(k_i) p_{ii}^+ + \sum_{i=1}^{M} \sum_{j=1,, j\neq i}^{M} \frac{q_i(k_i+1)}{q_j(k_j)} \gamma_i(k_i+1) p_{ij}^+ u(k_j).$$

After some transformations of the right part with combination with (20) and (21) we obtain

$$B = \sum_{j=1}^{M} (\lambda_j^- u(k_j) + \mu_j^+(k_j) + \gamma_j(k_j)). \qquad (29)$$

Finally, introducing

$$C = \Lambda_0 + \sum_{i=1}^{M} \lambda_i^- (1 - u(k_i)) \qquad (30)$$

we represent the right part of equalities (27) as $A + B + C$.

Using (28) - (30), where Λ_0, λ_i^+ and λ_i^- are determined by relations (21) and (22), we represent the right part of the equality (27) in the following form:

$$A + B + C = \sum_{j=1}^{M} \mu_j^-(n_j) + \sum_{j=1}^{M} (\lambda_j^- u(k_j) + \mu_j^+(k_j) + \gamma_j(k_j)) + \Lambda_0 + \sum_{i=1}^{M} \lambda_i^- - \sum_{i=1}^{M} \lambda_i^- u(k_i) = \sum_{j=1}^{M} \mu_j^-(n_j) + \sum_{j=1}^{M} (\mu_j^+(k_j) + \gamma_j(k_j)) + \Lambda_0 + \sum_{i=1}^{M} \lambda_i^- = \lambda_0^+ + \lambda_0^- + \sum_{j=1}^{M} \mu_j^-(n_j) + \sum_{j=1}^{M} (\mu_j^+(k_j) + \gamma_j(k_j)).$$

Thus the substitution of (25), (26) into the system of equations (3), (19), for all $(\vec{k}, \vec{n}) \in \mathcal{X}$, leads to a system of identities. Therefore, the expressions (25), (26) give a solution of the equilibrium system of equations (3), (19) which under the assumptions of the theorem is positive and bounded. As a consequence of this result the process $\{X(t), \quad t \geq 0\}$ is ergodic, thus the theorem is proved.

5 Conclusions

G-networks provide a versatile class to model complex systems in various applications fields such as computer network and telecommunication systems. In this paper we introduced the impatient service in G-netwowks with signals. Product form results for the case in which positive customers are processed by a single server at every node and for a symmetrical network have been obtained.

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