A CHARACTERIZATION OF PRODUCT-FORM STATIONARY DISTRIBUTIONS FOR QUEUEING SYSTEMS IN RANDOM ENVIRONMENT

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Abstract: We consider a general model for continuous-time Markov chains representing queueing systems in random environment. First we study the relationship between its equilibrium distribution and the Palm (or embedded) distributions at certain environmental change epochs. The results enable one to obtain the equilibrium distribution of the continuous-time model in terms of the equilibrium distribution of a discrete-time process. This is useful in simulation studies where one can extract information for the continuous-time model by recording the state of the system only in environmental change epochs. Then, we obtain necessary and sufficient conditions that ensure a product-form stationary distribution for the model. The related topics of partial balance and the ESTA (Events See Time Averages) property are also studied. As an illustration, we apply the results to study the stationary distributions of Jackson networks in random environment. For models that do not satisfy the product-form conditions, we develop a product-form approximation, which is proved to be very good for models evolving in a slowly changing random environment. We justify this fact by proving an explicit error bound for this approximation.
Keywords: queueing; Markov chains; random environment; stationary distribution; product form

1. INTRODUCTION

In many applications of queueing we find systems that evolve in random environment. The random environment may model the irregularity of the arrival process (for example when there are rush-hour phenomena or a periodically changing arrival stream), the irregularity of the service mechanism (due to servers' breakdowns, servers' vacations, availability of resources etc.) or both. Many authors have studied the properties of such models (see e.g. [Neuts, 1981], [Gaver at al., 1984], [O' Cinneide and Purdue, 1986], [Gelenbe and Rosenberg, 1990], [Falin, 1996] etc.). The reported results concern either qualitative properties or computational issues. The focus of the present work is on a computational issue and more specifically on the problem of computing the stationary distribution of a Markovian queueing system in random environment. Several authors have considered the same problem using various approaches. Matrix-analytic, transform methods and the so-called eigenvalue or spectral decomposition method have been used extensively. However, although in many cases the above methods give very satisfactory results, their implementation is computationally very demanding. The reason is that they require strong computational power to perform a great number of matrix operations. As the environmental state space grows large, the numerical complexity of the underlying algorithms increases rapidly and the efficient implementation of these methods becomes very difficult.

To avoid the computational burden of the above methods, several authors have tried to identify some categories of models for which the stationary distributions assume a simple product form. Although product-form stationary distributions and the related phenomenon of partial balance have been extensively studied within the framework of queueing networks (see [Kelly, 1979], [Gelenbe and Pujolle, 1998], [Chao et al., 1999] and [Serfozo, 1999]), there are only few papers that apply these ideas to queueing systems in random environment. More specifically [Sztrik, 1987], [Zhu, 1994], [Falin, 1996] and [Tsitsiashvili et al., 2002] have identified conditions that ensure product-form stationary distributions for several concrete classes of queueing systems in random environment. In the present paper we study the same problem within a general framework and we prove necessary and sufficient conditions for product-form. Moreover, whenever these conditions fail, we develop a product-form approximation, which is very good for queueing systems evolving in a slowly changing environment. Apart from an intuitive justification of this approximation, we also give an explicit error bound that shows when this approximation is legitimate and in what extent.

When the analytical study of a model seems too difficult or even impossible, the alternative is to perform a simulation study. Such a study requires the recording of the state of the system to compute sample-path averages and percentages that are used as estimates for the performance measures (expected values and probabilities respectively) of interest. It is much more convenient to record the state of the model at certain discrete transition epochs than in continuous time. To this end we study the relationship between the equilibrium distribution of a given Markovian queueing model in random environment and the equilibrium distributions of its embedded chains at environmental change epochs. The results enable us to obtain the equilibrium distribution of the continuous-time model in terms of the equilibrium distribution of a discrete-time process which can be estimated conveniently using simulation methods. We also study the associated Events See Time Averages (ESTA) property for the class of Markovian queueing systems in random environment.

To be concrete, we now define a general structure for a continuous-time Markov chain in a random environment. The model is an ergodic (i.e. irreducible and positive recurrent) Markov chain $\{(E(t), X(t)) : t \geq 0\}$ with state space $\mathbf{E}\times\mathbf{X}$, where $\{E(t)\}$, $\{X(t)\}$ represent the random environment and the queueing process of interest respectively. We assume that $\{E(t)\}$ jumps from state to state according to an ergodic continuous-time Markov chain with transition rate matrix $\mathbf{Q}_E = (q_E(e,e') : e, e' \in \mathbf{E})$. In the meantime between two successive environmental transitions, the process $\{X(t)\}$ is governed by a transition matrix $\mathbf{Q}_{X|E}(e) = (q_{X|E}(x,x'|e): \quad x, x' \in \mathbf{X})$ of an irreducible Markov chain on $\mathbf{X}$, where e is the current environmental state. More specifically the transition rates $q((e,x),(e',x'))$ of $\{(E(t), X(t))\}$ are given by

$$q((e,x),(e',x')) = \begin{cases} q_{X|E}(x,x'|e), & \text{if } e'=e, x'\neq x \\ q_E(e,e'), & \text{if } e'\neq e, x'=x. \end{cases} \qquad (1)$$

Let $\bar{p} = (p(e,x) : e \in \mathbf{E}, x \in \mathbf{X})$ be the joint equilibrium (stationary) distribution of $\{(E(t), X(t))\}$ and denote by $\bar{p}_E = (p_E(e) : e \in \mathbf{E})$, $\bar{p}_X = (p_X(x) : x \in \mathbf{X})$ its marginal distributions. The (full) balance equations are

$$p(e,x)\Big(\sum_{e'\neq e} q_E(e,e') + \sum_{x'\neq x} q_{X|E}(x,x'|e)\Big) = \sum_{e'\neq e} p(e',x)q_E(e',e) + \sum_{x'\neq x} p(e,x')q_{X|E}(x',x|e), \quad e \in \mathbf{E}, x \in \mathbf{X}. \qquad (2)$$

By summing these equations over x for every fixed environmental state e we obtain after some easy manipulations that

$$p_E(e)\sum_{e'\neq e} q_E(e,e') = \sum_{e'\neq e} p_E(e')q_E(e',e), \quad e \in \mathbf{E}. \qquad (3)$$

Hence, the marginal distribution $\bar{p}_E = (p_E(e))$ is the equilibrium distribution of the Markov chain with transition rate matrix $\mathbf{Q}_E = (q_E(e,e'))$.

Let $\mathbf{P}^{(t)}_{X|E}(e) = (p^{(t)}_{X|E}(x,x'|e) : x, x' \in \mathbf{X})$ be the transition probability matrix at time t for the Markov chain with transition rate matrix $\mathbf{Q}_{X|E}(e)$ and $\bar{p}_{X|E}(e) = (p_{X|E}(x|e) : x \in \mathbf{X})$ be its equilibrium distribution (in the ergodic case in which it exists and is unique). We are interested in determining $\bar{p}, \bar{p}_X$ and in examining their relationships with the transition rate matrices $\mathbf{Q}_E$ and $\mathbf{Q}_{X|E}(e)$, $e \in \mathbf{E}$.

We are also interested in studying the Palm (or embedded) distributions of $\{X(t)\}$ just after (or before) certain environmental transitions. For every $e \in \mathbf{E}$ let $X_{a(e)}(n)$ be the state of $\{X(t)\}$ just after the n-th environmental arrival to e and $X_{d(e)}(n)$ be the state of $\{X(t)\}$ just before the n-th environmental departure from e. Denote the stationary distributions of $\{X_{a(e)}(n)\}$ and $\{X_{d(e)}(n)\}$ by $\bar{p}_{a(e)} = (p_{a(e)}(x) : x \in \mathbf{X})$ and $\bar{p}_{d(e)} = (p_{d(e)}(x) : x \in \mathbf{X})$ respectively. Moreover, define the quantities $q_{X|E}(x|e) = \sum_{x'\neq x} q_{X|E}(x,x'|e)$ and $q_E(e) = \sum_{e'\neq e} q_E(e,e')$. Because of (3) we have that

$$\sum_y \sum_{e'\neq e} p(e',y)q_E(e',e) = p_E(e)q_E(e).$$

Since the rate from state (e,x) to (e',x') is $p(e,x)q((e,x),(e',x'))$ we have that

$$p_{a(e)}(x) = \frac{\sum_{e'\neq e} p(e',x)q_E(e',e)}{\sum_y \sum_{e'\neq e} p(e',y)q_E(e',e)} = \frac{\sum_{e'\neq e} p(e',x)q_E(e',e)}{p_E(e)q_E(e)}, \quad x \in \mathbf{X} \qquad (4)$$

and

$$p_{d(e)}(x) = \frac{\sum_{e'\neq e} p(e,x)q_E(e,e')}{\sum_y \sum_{e'\neq e} p(e,y)q_E(e,e')} = \frac{p(e,x)}{p_E(e)}, \quad x \in \mathbf{X}. \qquad (5)$$

It is known that the Palm distributions of a process that correspond to different sets of transitions do not coincide with each other nor do they coincide with the stationary distribution of the process in general. In such cases it is important to study the relationships of these distributions and also to find conditions under which they do coincide (Events See Time Averages (ESTA) property).

2. CHARACTERIZATION OF PRODUCT-FORM DISTRIBUTIONS

Equations (2) are decomposed to the following partial balance equations that are not satisfied by $\bar{p}$ in general:

$$p(e,x)\sum_{e'\neq e} q_E(e,e') = \sum_{e'\neq e} p(e',x) q_E(e',e), \quad e\in \mathbf{E}, \quad x\in \mathbf{X} \tag{6}$$

and

$$p(e,x)\sum_{x'\neq x} q_{X|E}(x,x'|e) = \sum_{x'\neq x} p(e,x') q_{X|E}(x',x\,|\,e), \; e\in \mathbf{E},\; x\in \mathbf{X}. \tag{7}$$

The phenomenon of partial balance and its implications have been extensively studied in the literature (see e.g. [Kelly, 1979]). It has been generally noted that the presence of partial balance facilitates the study of a given model. First, it implies the equality of the Palm distributions at certain event epochs (see e.g. [Kelly, 1979 Ch. 9] and [Fakinos and Economou, 1998]. Second, under certain additional conditions, it implies that the stationary distribution assumes a certain product form. To characterize the phenomenon of partial balance and product-form for the general model (1), we first prove the following result that gives the stationary distribution $\bar{p}$ in terms of the Palm distributions $\bar{p}_{a(e)}$, $e\in \mathbf{E}$. This result is also of independent interest as it can be used in simulation studies. As we have already explained in the introduction, it is convenient first to obtain estimates for $\bar{p}_{a(e)}$, $e\in \mathbf{E}$ using a simulation methodology and then compute $p(e,x)$, $e\in \mathbf{E}$, $x\in \mathbf{X}$, using an analytic formula.

Theorem 1 *(Inversion formula) Given the Palm distributions $\bar{p}_{a(e)}$, the stationary distribution $\bar{p}$ can be computed by*

$$p(e,x) = p_E(e) q_E(e) \int_0^\infty e^{-q_E(e)t} p_{a(e)}^{(t)}(x)\,dt \tag{8}$$

where $\bar{p}_{a(e)}^{(t)} = (p_{a(e)}^{(t)}(x) : x\in \mathbf{X})$ is the transient probability distribution at time t of a Markov chain with initial distribution $\bar{p}_{a(e)}$ and transition rate matrix $\mathbf{Q}_{X|E}(e)$.

Proof. By the definition of $\bar{p}_{a(e)}^{(t)}$ and the relation (4) we have that

$$p_{a(e)}^{(t)}(x) = \sum_{x'} p_{a(e)}(x') p_{X|E}^{(t)}(x',x\,|\,e) = \frac{\sum_{x'}\sum_{e'\neq e} p(e',x') q_E(e',e) p_{X|E}^{(t)}(x',x\,|\,e)}{p_E(e) q_E(e)}. \tag{9}$$

Hence

$$p_E(e) q_E(e) \int_0^\infty e^{-q_E(e)t} p_{a(e)}^{(t)}(x)\,dt = \sum_{x'}\sum_{e'\neq e} p(e',x') q_E(e',e) \int_0^\infty e^{-q_E(e)t} p_{X|E}^{(t)}(x',x\,|\,e)\,dt,$$
$$e\in \mathbf{E}, \quad x\in \mathbf{X}. \tag{10}$$

Using the balance equations (2), equation (10) is written as

$$p_E(e) q_E(e) \int_0^\infty e^{-q_E(e)t} p_{a(e)}^{(t)}(x)\,dt = A(e,x) + B(e,x) - C(e,x), \quad e\in \mathbf{E}, \quad x\in \mathbf{X}, \tag{11}$$

where

$$A(e,x) = \sum_{x'} p(e,x') q_E(e) \int_0^\infty e^{-q_E(e)t} p_{X|E}^{(t)}(x',x\,|\,e)\,dt,$$

$$B(e,x) = \sum_{x'} p(e,x') q_{X|E}(x'|e) \cdot \int_0^\infty e^{-q_E(e)t} p_{X|E}^{(t)}(x',x\,|\,e)\,dt,$$

$$C(e,x) = \sum_{x'}\sum_{y\neq x'} p(e,y) q_{X|E}(y,x'|e) \cdot \int_0^\infty e^{-q_E(e)t} p_{X|E}^{(t)}(x',x\,|\,e)\,dt.$$

By the Chapman-Kolmogorov equations we have that

$$\sum_{x'\neq y} q_{X|E}(y,x'|e) p_{X|E}^{(t)}(x',x\,|\,e) = \tfrac{d}{dt} p_{X|E}^{(t)}(y,x\,|\,e) + q_{X|E}(y\,|\,e) p_{X|E}^{(t)}(y,x\,|\,e),$$

and therefore we obtain that

$$C(e,x) = D(e,x) + E(e,x)$$

where

$$D(e,x) = \sum_{y} p(e,y) \int_0^\infty e^{-q_E(e)t} \tfrac{d}{dt} p_{X|E}^{(t)}(y,x\,|\,e)\,dt,$$

$$E(e,x) = \sum_{y} p(e,y) q_{X|E}(y\,|\,e) \cdot \int_0^\infty e^{-q_E(e)t} p_{X|E}^{(t)}(y,x\,|\,e)\,dt.$$

We note that $E(e,x)=B(e,x)$ and (11) assumes the form

$$p_E(e)q_E(e)\int_0^\infty e^{-q_E(e)t}p_{a(e)}^{(t)}(x)dt$$
$$=A(e,x)-D(e,x),\quad e\in\mathbf{E},\ x\in\mathbf{X}. \qquad (12)$$

Using integration by parts, we have that

$$D(e,x)=\sum_y p(e,y)\left[e^{-q_E(e)t}p_{X|E}^{(t)}(y,x\mid e)\right]_{t=0}^\infty$$
$$+\sum_y p(e,y)\int_0^\infty q_E(e)e^{-q_E(e)t}p_{X|E}^{(t)}(y,x\mid e)dt$$
$$=-p(e,x)+A(e,x), \qquad (13)$$

since $\lim_{t\to\infty}e^{-q_E(e)t}p_{X|E}^{(t)}(y,x\mid e)=0$, while $p_{X|E}^{(0)}(y,x\mid e)=1_{\{x\}}(y)$.

We plug (13) into (12) and we conclude (8).

We are now in position to investigate the phenomenon of partial balance within the framework of our model and to derive conditions that ensure a product-form stationary distribution.

Theorem 2 *For the general model with transition rates given by (1) the following are equivalent:*
(i) The Palm distributions $\bar{p}_{a(e)}$ *and* $\bar{p}_{d(e)}$ *coincide for every* $e\in\mathbf{E}$.
(ii) The stationary $\bar{p}$ *satisfies the partial balance equations (6).*
(iii) The stationary $\bar{p}$ *satisfies the partial balance equations (7).*
If moreover the transition matrices $\mathbf{Q}_{X|E}(e)$ *are ergodic with stationary distributions* $\bar{p}_{X|E}(e)$, $e\in\mathbf{E}$ *then (i)-(iii) are also equivalent to:*
(iv) The distributions $\bar{p}_{d(e)}$ *and* $\bar{p}_{X|E}(e)$ *coincide for every* $e\in\mathbf{E}$.
(v) The distributions $\bar{p}_{a(e)}$ *and* $\bar{p}_{X|E}(e)$ *coincide for every* $e\in\mathbf{E}$.
(vi) The stationary distribution $\bar{p}$ *is given by the product-form formula*

$$p(e,x)=p_E(e)p_{X|E}(x\mid e),\ e\in\mathbf{E},\ x\in\mathbf{X}. \qquad (14)$$

Proof. $(i)\Leftrightarrow(ii)$ The probabilities $p_{a(e)}(x)$ and $p_{d(e)}(x)$ given by (4) and (5) are respectively equal to the right and the left side of the partial balance equations (6) divided by $p_E(e)q_E(e)$.

$(ii)\Leftrightarrow(iii)$ Immediate, in light of the full balance equations (2).

$(iii)\Rightarrow(vi)$ Consider a fixed $e\in\mathbf{E}$. Because of (7) we have that the vector $(p(e,x):x\in\mathbf{X})$ satisfies the balance equations of the Markov chain with transition rate matrix $\mathbf{Q}_{X|E}(e)$. Due the ergodicity of $\mathbf{Q}_{X|E}(e)$, we have that $(p(e,x):x\in\mathbf{X})$ is a scalar multiple of the stationary distribution $\bar{p}_{X|E}(e)$, i.e. $p(e,x)=c(e)p_{X|E}(x\mid e)$, $x\in\mathbf{X}$. By summing over x we obtain $c(e)=p_E(e)$; hence $p(e,x)$ assumes the form (14).

$(iv)\Rightarrow(vi)$ Immediate using (5).

$(v)\Rightarrow(vi)$ Since $\bar{p}_{a(e)}$ and $\bar{p}_{X|E}(e)$ coincide, the Inversion formula (8) assumes the form

$$p(e,x)=p_E(e)q_E(e)$$
$$\cdot\int_0^\infty e^{-q_E(e)t}p_{X|E}^{(t)}(x\mid e)dt,\ e\in\mathbf{E},\ x\in\mathbf{X}. \qquad (15)$$

But $(p_{X|E}(x\mid e):x\in\mathbf{X})$ is the stationary distribution of $\mathbf{Q}_{X|E}(e)$ and we have that $p_{X|E}^{(t)}(x\mid e)=p_{X|E}(x\mid e)$, $t\geq 0$, $e\in\mathbf{E}$, $x\in\mathbf{X}$. Equation (15) is reduced to (14).

$(vi)\Rightarrow(iii),(iv),(v)$ If the stationary distribution $\bar{p}$ is given by (14) then we have obviously that the partial balance equations (7) hold, i.e. (iii) is valid. Moreover, by (5) we have that $p_{d(e)}(x)=p_{X|E}(x\mid e)$, $x\in\mathbf{X}$, i.e. (iv) is valid. We have also that (i) holds because of the implication $(iii)\Rightarrow(i)$ that has already been proved. Hence $p_{a(e)}(x)=p_{d(e)}(x)=p_{X|E}(x\mid e)$, $e\in\mathbf{E}$, $x\in\mathbf{X}$, i.e. (v) is valid.

The above result characterizes completely the partial balance, the product form and the ESTA properties for the model (1). However, we see that the conditions that imply a product-form stationary distribution are very restrictive. We now consider a 'perturbed' model that has always a product-form distribution. We have the following.

Theorem 3 *Consider a continuous-time Markov chain with state-space* $\mathbf{E}\times\mathbf{X}$ *and transition rates*

$$\tilde{q}((e,x),(e',x'))=\begin{cases}q_{X|E}(x,x'\mid e), & \text{if } e'=e, x'\neq x\\ q_E(e,e')p_{X|E}(x'\mid e'), & \text{if } e'\neq e, x'\in\mathbf{X}\end{cases} \qquad (16)$$

where $q_{X|E}(x,x'\mid e)$, $q_E(e,e')$ *and* $p_{X|E}(x\mid e)$ *are the same as in the model* (1). *Then its stationary distribution is given by the product-form formula*

$$\tilde{p}(e,x)=p_E(e)p_{X|E}(x\mid e),\ e\in\mathbf{E},\ x\in\mathbf{X}. \qquad (17)$$

Proof. The balance equations of the model are

$$\tilde{p}(e,x)(\sum_{e'\neq e}\sum_{x\in\mathbf{X}}q_E(e,e')p_{X|E}(x'\mid e')$$

$$+\sum_{x'\neq x} q_{X|E}(x,x'|e))$$
$$=\sum_{e'\neq e}\sum_{x'\in \mathbf{X}}\tilde{p}(e',x')q_E(e',e)p_{X|E}(x|e)$$
$$+\sum_{x'\neq x}\tilde{p}(e,x')q_{X|E}(x',x|e). \qquad (18)$$

By direct substitution we see that the distribution $(\tilde{p}(e,x): e\in \mathbf{E}, x\in \mathbf{X})$ satisfies the equations (18); hence it is the stationary distribution of the model.

For an intuitive understanding of Theorem 3, note that the transition rates (16) imply that in the perturbed model the process of interest $\{X(t)\}$ starts anew in equilibrium after every environmental change. Thus, we can think of $\{X(t)\}$ as the recorded value of a variable at time t of an experiment that is performed concurrently in $|\mathbf{E}|$ positions under various environmental conditions, where the observer moves from position to position according to $\{E(t)\}$. If the various evolutions of the experiment at the $|\mathbf{E}|$ positions are independent and in equilibrium and the observer does not influence the experiment then the transition rates of $\{(E(t),X(t))\}$ are of the form (16). Moreover, because of the independence assumptions, we expect the stationary probability $\tilde{p}(e,x)$ of seeing the environment at state e and the process of interest at state x to be given by the product-form formula (17).

Whenever the transition rates $q_E(e,e')$ of the environmental process $\{E(t)\}$ are small, the rates $\tilde{q}((e,x),(e',x'))$ of the perturbed model (16) are very close to the rates $q((e,x),(e',x'))$ of the original model (1). Hence the stationary distributions of the two models are expected to be also very close to each other and we conclude that the product-form distribution of the modified model (16) is indeed a good approximation for the stationary distribution of the original model (1). Thus, this product-form distribution is a legitimate approximation for queueing systems evolving in a slowly changing environment. We aim at providing error bounds for this approximation. Recall that in the case of finite Markov chains the convergence of the transient distributions to the stationary distribution is uniform and geometric. More precisely, for an irreducible and finite continuous-time Markov chain with transition matrices $\mathbf{P}^{(t)}=(p^{(t)}(x,y))$ and stationary distribution $\bar{p}=(p(y))$ there exist constants $C,a>0$ such that

$$\sup_{x'}\sum_x |p^{(t)}(x',x)-p(x)|<Ce^{-at},\ t>0. \qquad (19)$$

The quantities C,a are called the convergence-to-stationarity parameters of the Markov chain and can be computed efficiently in a number of cases (see e.g. [Bremaud, 1999 Chapter 6]. Using the convergence-to-stationarity parameters of Markov chains, we can provide easily computable error bounds for the approximation of the stationary distribution $(p(e,x))$ of a given model with transition rates (1) by the product-form distribution $(p_E(e)p_{X|E}(x|e))$. We have the following:

Theorem 4 Consider a continuous-time Markov chain in random environment $\{(E(t),X(t))\}$ with transition rates (1) and stationary distribution $(p(e,x))$. Suppose that the state space $\mathbf{X}$ of $\{X(t)\}$ is finite and therefore for every $e\in\mathbf{E}$ there exists parameters $C_e, a_e>0$ such that

$$\sup_{x'}\sum_x |p^{(t)}_{X|E}(x',x|e)-p_{X|E}(x|e)|\le C_e\exp(-a_e t),\quad t>0. \qquad (20)$$

Then the following error bound hold:

$$\sum_x |p(e,x)-p_E(e)p_{X|E}(x|e)|\le \frac{C_e p_E(e)q_E(e)}{q_E(e)+a_e}. \qquad (21)$$

Proof. The Inversion formula (8) and equation (10) imply that

$$p(e,x)=\sum_{x'}\sum_{e'\neq e}p(e',x')q_E(e',e)$$
$$\cdot\int_0^\infty e^{-q_E(e)t}p^{(t)}_{X|E}(x',x|e)dt,\quad e\in\mathbf{E}, x\in\mathbf{X}. \qquad (22)$$

A similar expression is valid for the product-form approximating distribution $(p_E(e)p_{X|E}(x|e))$. More specifically, we have

$$p_E(e)p_{X|E}(x|e)=\sum_{x'}\sum_{e'\neq e}p(e',x')q_E(e',e)$$
$$\cdot\int_0^\infty e^{-q_E(e)t}p_{X|E}(x|e)dt,\quad e\in\mathbf{E}, x\in\mathbf{X}. \qquad (23)$$

Indeed by (3) we have that

$$\sum_{x'}\sum_{e'\neq e}p(e',x')q_E(e',e)=p_E(e)q_E(e),\quad e\in\mathbf{E}$$

and (23) follows.

By subtracting (23) from (22) we obtain

$$p(e,x)-p_E(e)p_{X|E}(x|e)=\sum_{x'}\sum_{e'\neq e}p(e',x')q_E(e',e)$$
$$\cdot\int_0^\infty e^{-q_E(e)t}(p^{(t)}_{X|E}(x',x|e)-p_{X|E}(x|e))dt,$$
$$e\in\mathbf{E}, x\in\mathbf{X}. \qquad (24)$$

Taking absolute values in (24) and summing over all x for a fixed e we obtain

$$\sum_x |p(e,x)-p_E(e)p_{X|E}(x|e)|$$

$$\leq \sum_{x'} \sum_{e' \neq e} p(e', x') q_E(e', e)$$

$$\cdot \int_0^\infty e^{-q_E(e)t} \sum_x | p_{X|E}^{(t)}(x', x \mid e) - p_{X|E}(x \mid e) | dt, \; e \in \mathbf{E}.$$

Using (20) we obtain that

$$\sum_x | p(e,x) - p_E(e) p_{X|E}(x \mid e) |$$

$$\leq \sum_{x'} \sum_{e' \neq e} p(e', x') q_E(e', e) \int_0^\infty e^{-q_E(e)t} C_e e^{-a_e t} dt$$

$$\leq \sum_{x'} \sum_{e' \neq e} p(e', x') q_E(e', e) \frac{C_e}{q_E(e) + a_e}, \; e \in \mathbf{E}. \quad (25)$$

But by (3) we have that

$$\sum_{x'} \sum_{e' \neq e} p(e', x') q_E(e', e) = p_E(e) q_E(e), \;\; e \in \mathbf{E}$$

and (25) reduces to (21).

The quality of the bound (21) depends on the rate (speed) of the environmental changes. For environmental states with $q_E(e)$ near to 0 (i.e. when the environment evolves very slowly) we have that the bound tends to 0 and the product-form distribution $(p_E(e) p_{X|E}(x \mid e))$ is indeed a good approximation for the stationary distribution $(p(e,x))$. The same is true when a_e is very large with comparison to $q_E(e)$ (i.e. when the Markov chain with transition rate matrix $\mathbf{Q}_{X|E}(e)$ converges to equilibrium at a high rate).

Unfortunately, the bound suggested by Theorem 4 is applicable only for finite Markov chains in random environment. For infinite Markov chains a more involved approach should be used. More specifically, in the context of concrete infinite models we can estimate the error of the approximation using the results obtained by van Dijk (1992). These results provide error bounds for the approximation of the stationary distribution of a given model by the stationary distribution of a perturbed model in terms of the differences of their transition rates, using a Markov reward approach.

3. AN APPLICATION TO JACKSON NETWORKS IN RANDOM ENVIRONMENT

As an illustration of the main result, we present its application in the study of Jackson networks in random environment. A Jackson network in random environment is a continuous-time Markov chain on $\mathbf{E} \times \mathbf{Z}_+^J$ with transition rates given by (1) and matrices $\mathbf{Q}_{X|E}(e), \;\; e \in \mathbf{E}$ corresponding to Jackson networks, i.e.

$$q_{X|E}(\bar{x}, \bar{x}' \mid e)$$

$$= \begin{cases} l(e) p_{0j}(e) & \text{if } \bar{x}' = \bar{x} + \bar{e}_j \\ m_i(x_i \mid e) p_{ij}(e) & \text{if } \bar{x}' = \bar{x} - \bar{e}_i + \bar{e}_j \\ m_i(x_i \mid e) p_{i0}(e) & \text{if } \bar{x}' = \bar{x} - \bar{e}_i \\ 0 & \text{otherwise,} \end{cases} \quad (26)$$

where by $\bar{x} = (x_1, x_2, ..., x_J)$ we denote a generic state of the network representing the queue lengths at the J stations and $\bar{e}_j$ is the j-th unit vector with J components (with 1 in the j-th position and 0 elsewhere).

Therefore, in any time interval during which the environmental process $\{E(t)\}$ is in state e, the network operates as follows: Customers arrive at the network according to a Poisson process with rate $l(e)$. An arriving customer goes to the j-th station with probability $p_{0j}(e)$ ($j = 1,2,\ldots,J$). The service at the i-th station of the network is offered at exponential rate $m_i(x_i \mid e)$ which depends on the number x_i of the present customers at that same station ($i = 1,2,..., J$). Upon completing service at the i-th station, a customer is routed to station j with probability $p_{ij}(e)$ or leaves the network with probability $p_{i0}(e)$, ($i, j = 1,2,\ldots,J$). For every fixed $e \in \mathbf{E}$, the discrete-time Markov chain with transition probabilities $p_{ij}(e)$ ($i, j = 1,2,\ldots,J$) is supposed to be irreducible. This implies that the traffic equations

$$a_j(e) = l(e) p_{0j}(e) + \sum_{i=1}^J a_i(e) p_{ij}(e), \quad j = 1,2,\ldots,J \quad (27)$$

have a unique positive solution $\bar{a}(e) = (a_1(e), a_2(e), ..., a_J(e))$. Moreover, all the arrival and service processes are assumed independent. For a fixed e, the Markov chain representing a Jackson network with rates given by (26) is positive recurrent if and only if

$$B_j^{-1}(e) = 1 + \sum_{x_j=1}^\infty \frac{a_j(e)^{x_j}}{m_j(1 \mid e) m_j(2 \mid e) \ldots m_j(x_j \mid e)} < \infty, \quad j = 1,2,\ldots,J. \quad (28)$$

The stationary distribution is then given by

$$p_{X|E}(\bar{x} \mid e) = \prod_{j=1}^J B_j(e) \frac{a_j(e)^{x_j}}{m_j(1 \mid e) m_j(2 \mid e) \ldots m_j(x_j \mid e)}. \quad (29)$$

[Zhu, 1994] proved from scratch a sufficient condition for product form. Using Theorem 2 we can easily show the necessity and the sufficiency of that condition for product form.

Corollary 4 *Let* $\{E(t), \overline{X}(t)\}$ *be a Jackson network in random environment with transition rates given by (1) and (26). For every* $e \in \mathbf{E}$, *let* $\overline{a}(e) = (a_1(e), a_2(e), ..., a_J(e))$ *be the unique solution of the system of equations (27) and assume that the stability condition (28) holds. The following are equivalent:*

(i) $a_j(e)/m_j(x_j \mid e)$ *is independent of* e, *for all* $j = 1, 2, \dots, J$ *and* $x_j \geq 1$.

(ii) *The stationary distribution* $\overline{p}$ *is given by the product-form formula*

$$p(e, \overline{x}) = p_E(e) \prod_{j=1}^{J} B_j(e) \frac{a_j(e)^{x_j}}{m_j(1 \mid e) m_j(2 \mid e) ... m_j(x_j \mid e)}, \quad e \in \mathbf{E}, \quad \overline{x} \in \mathbf{Z}_+^J, \tag{30}$$

where $(p_E(e) : e \in \mathbf{E})$ *is the stationary distribution of a Markov chain with transition rates* $(q_E(e, e'))$ *and* $B_j(e)$ *are given by (28). The equivalent conditions (i)-(vi) of Theorem 2 hold.*

Proof. $(i) \Rightarrow (ii)$ Suppose that (i) holds. Then by direct substitution we can show that the distribution given by (30) satisfies the balance equations (2). Indeed, using the fact that $(p_E(e) : e \in \mathbf{E})$ and $(p_{X|E}(\overline{x} \mid e) : x \in \mathbf{Z}_+^J)$ are the stationary distributions of $\mathbf{Q}_E$ and $\mathbf{Q}_{X|E}(e)$ respectively, we have that

$$\begin{aligned} p_E(e) p_{X|E}(\overline{x} \mid e) & \cdot \left(\sum_{e' \neq e} q_E(e, e') + \sum_{x' \neq x} q_{X|E}(\overline{x}, \overline{x}' \mid e)\right) \\ &= p_{X|E}(\overline{x} \mid e) \sum_{e' \neq e} p_E(e') q_E(e', e) \\ &\quad + p_E(e) \sum_{\overline{x}' \neq \overline{x}} p_{X|E}(\overline{x}' \mid e) q_{X|E}(\overline{x}', \overline{x} \mid e). \end{aligned} \tag{31}$$

But by condition (i) and (28) we obtain that $B_j^{-1}(e)$ is independent of e for all $j = 1, 2, \dots, J$. Hence by (29) we conclude that $p_{X|E}(\overline{x} \mid e)$ is independent of e for all $\overline{x}$. Then (31) assumes the form

$$\begin{aligned} p_E(e) p_{X|E}(\overline{x} \mid e) & \cdot \left(\sum_{e' \neq e} q_E(e, e') + \sum_{x' \neq x} q_{X|E}(\overline{x}, \overline{x}' \mid e)\right) \\ &= \sum_{e' \neq e} p_E(e') p_{X|E}(\overline{x} \mid e') q_E(e', e) \\ &\quad + p_E(e) \sum_{\overline{x}' \neq \overline{x}} p_{X|E}(\overline{x}' \mid e) q_{X|E}(\overline{x}', \overline{x} \mid e). \end{aligned}$$

i.e. the distribution $(p(e, x))$ given by (30) satisfies the balance equations (2).

$(ii) \Rightarrow (i)$ By Theorem 2 $(vi) \Rightarrow (ii)$ we have that for every $\overline{x}$ the vector $(p(e, \overline{x}) : e \in \mathbf{E})$ satisfies the balance equations for the process $\{E(t)\}$. Hence $(p(e, \overline{x}) : e \in \mathbf{E})$ is a scallar multiple of $(p_E(e) : e \in \mathbf{E})$ and we conclude that $p(e, \overline{x}) = f(\overline{x}) p_E(e)$, $e \in \mathbf{E}$, $\overline{x} \in \mathbf{Z}_+^J$. Then for every $j = 1, 2, \dots, J$ and $x_j \geq 1$ we have

$$\frac{a_j(e)}{m_j(x_j \mid e)} = \frac{p(e, \overline{x})}{p(e, \overline{x} - \overline{e}_j)} = \frac{f(\overline{x})}{f(\overline{x} - \overline{e}_j)},$$

i.e. $a_j(e)/m_j(x_j \mid e)$ is independent of e.

ACKNOWLEDGEMENT

The author thanks the support received from the research project ELKE/70/4/6415 of the University of Athens.

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Antonis Economou received the MA degree in Pure Mathematics (1994) from the University of California, Los Angeles and the MSc degree in Statistics and OR (1997) and the PhD degree in Mathematics (1998) from the University of Athens, Greece. During 1999-2001 he was visiting faculty member at the Applied Mathematics Department of the University of Crete. Since 2001 he has been a faculty member at the Mathematics Department of the University of Athens. His research interests include the performance evaluation and the control of queueing systems and computational and stochastic comparison problems for Markov chains.