ANALYSIS OF A QUEUE IN THE $BMAP/G/1/N$ SYSTEM

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Abstract: A single server queue with a finite buffer is analyzed. Input is described by the $BMAP$ (Batch Markovian Arrival Process). The disciplines of complete admission and complete rejection are dealt with. The stationary queue length distribution at service completion and arbitrary epochs is calculated. The loss probability is found and its dependence on the discipline of admission, correlation and variation of the $BMAP$, service time variation, buffer capacity and the load of the system is illustrated by means of numerical examples.
Keywords: Finite queue; Batch Markovian Arrival Process; complete admission; complete rejection; loss probability

INTRODUCTION

Investigation of finite capacity queues is important in many applied problems for correct design of a buffer and optimal buffer pool sharing. Extensive research in this field was done by P. Bocharov.

A very general $BMAP/SM/1/N$ model was investigated in [Dudin et al., 2002a]. Assumption that the input flow is the $BMAP$ allows to use this result for modelling modern telecommunication networks where the flows of information are correlated and so they cannot be well approximated in terms of the stationary Poisson process (even with use of Hurst parameter). Assumption that the type of the service process is SM (Semi-Markovian) allows to capture a possible correlation of successive service times. Advantage of the paper [Dudin et al., 2002a] comparing to earlier papers [Blondia, 1989] and [Dudin and Nishimura, 2000], besides consideration of the more general SM service process, consists of the following. In the paper [Blondia, 1989], direct solving of a finite set of equilibrium equations is performed. So, the existing specifics of the transition probability matrix is practically ignored. In the paper [Dudin and Nishimura, 2000], such a specifics is taken into account effectively. However, the elaborated algorithm for computing the stationary distribution of the system states is not numerically stable for large values of the buffer capacity N. The algorithm presented in [Dudin et al., 2002a] takes into account the special structure of the transition probability matrix of the embedded Markov chain and simultaneously is very stable numerically.

A shortage of the paper [Dudin et al., 2002a] consists of incomplete account of possible admission disciplines. It is known that due to the capacity limitation and batch arrivals, a situation can occur when the batch size exceeds the currently available capacity of the buffer. Three admission disciplines for such a situation are known in literature: partial admission (PA), when only a part of the batch corresponding to the number of free places in the buffer is allowed to join the system; complete admission (CA), when the whole batch is allowed to enter the system; complete rejection (CR), when the whole batch is rejected. Only the discipline of PA is well investigated for models with a general service time distribution. This stems from the fact that the technique of the embedded Markov chains, which is very effective in research, has a difficulty with arrivals accounting between the embedded epochs when the disciplines of CA and CR are applied. However, these disciplines are very important, e. g., in modelling in telecommunications. If a batch is interpreted as a set of packages belonging to one information unit, it does not make sense to allow partial admission of the packages. Thus, the discipline of CR or CA type should be selected. Technical realization of the CR admission discipline is very easy, but the loss probability is rather high. The discipline of CA type may provide, especially in case of long batches, much less value of this probability and higher throughput of the system. But this discipline suggests a presence of some additional place for admitting a whole group which can not be completely placed into the buffer. However, it is not a problem in many real life systems. E.g., if we model a computer system we can consider RAM (Random Access Memory) as a finite buffer. In case of necessity, the rest of processed information can be placed into extended or expanded memory. In case the finite buffer under investigation is a part of memory in some buffer-pool, it is normal that some additional part of the common buffer-pool can be temporarily used on demand.

Thus, the analysis given in [Dudin et al., 2002a] is incomplete. In this paper we supplement investigation of the $BMAP/G/1/N$ queue by considering the disciplines CR and CA.

THE MATHEMATICAL MODEL

We have a single server queue with a finite buffer. The capacity of the buffer is N. Customers arrive into the system according to the $BMAP$. The behavior of the $BMAP$ is defined by the underlying process ν_t, $t \geq 0$, which is an irreducible continuous-time Markov chain with the state space $\{0, 1, \ldots, W\}$. The customers arrive at the epochs when the chain ν_t, $t \geq 0$, makes transitions. The matrix D_k defines the transitions of the chain ν_t, $t \geq 0$, which are accompanied by arrival of a batch consisting of k customers, $k \geq 0$. Denote $D(z) = \sum\limits_{k=0}^{\infty} D_k z^k$, $|z| \leq 1$. The matrix $D(1)$ is the infinitesimal generator of the process ν_t, $t \geq 0$. The vector $\boldsymbol{\theta}$ of steady state distribution of the chain ν_t, $t \geq 0$, satisfies the system $\boldsymbol{\theta} D(1) = \mathbf{0}$, $\boldsymbol{\theta}\mathbf{e} = 1$. Here $\mathbf{e}$ is a column vector consisting of all ones, $\mathbf{0}$ is a row vector consisting of all zeroes. The average intensity λ of the $BMAP$ (fundamental rate) is calculated as

$$\lambda = \boldsymbol{\theta} D'(1)\mathbf{e}$$

, and the intensity λ_g of group arrivals is defined as

$$\lambda_g = \boldsymbol{\theta}(-D_0)\mathbf{e}.$$

The variance v of intervals between group arrivals is calculated as

$$v = 2{\lambda_g}^{-1}\boldsymbol{\theta}(-D_0)^{-1}\mathbf{e} - {\lambda_g}^{-2},$$

while the correlation coefficient c_{cor} and the squared variation coefficient c_{var}^2 of intervals between successive group arrivals are given by

$$c_{cor} = ({\lambda_g}^{-1}\boldsymbol{\theta}(-D_0)\times \\ \times(D(1) - D_0)(-D_0)^{-1}\mathbf{e} - {\lambda_g}^{-2})/v,$$

$$c_{var}^2 = v\lambda_g^2.$$

The $BMAP$ is a popular descriptor of flows in modern telecommunication networks. It allows to capture their bursty correlated nature. The $BMAP$ generalizes such known flows as the stationary Poisson Process, phase type (PH) input process, Markov Modulated Poisson Process ($MMPP$), Interrupted Poisson Process (IPP), etc. So, the models of queues with the $BMAP$ are investigated in literature intensively. For more details about the $BMAP$ and related research see [Lucantoni, 1991], [Chakravarthy, 2001].

The service time is characterized by the distribution function $B(t)$ and the mean service time $b_1 = \int\limits_0^{\infty} t dB(t)$. As it was mentioned above, the present model is less general comparing to the one in [Dudin et al., 2002a] where the SM service is considered. The extension of the presented below results to the case of SM service is straightforward if we take into account the technical aspects presented, e. g., in [Dudin and Karolik, 2001].

THE EMBEDDED PROCESS

The process of interest is the number i_t of customers in the system at the epoch t, $t > 0$. The state space of this process is $\{0, 1, \ldots, N+1\}$ for the disciplines of PA and CR. The state space is unlimited, in general, in case of CA discipline. This process is non-Markovian. So, we first investigate the embedded process i_{t_n+0}, where t_n is the n-th service completion epoch, $n \geq 1$. For the simplicity of denotations in the sequel we use denotation $i_n = i_{t_n+0}$, $n \geq 1$. This process is non-Markovian as well. But the two-dimensional process $\xi_n = \{i_n, \nu_n\}$, $n \geq 1$, where $\nu_n = \nu_{t_n}$ is a two-dimensional Markov chain.

A general stable algorithm for calculating the stationary state distribution of multi-dimensional Markov chains having the structure of the transition probability matrix similar to the one for the Markov chain $\xi_n, n \geq 0$, is presented in [Klimenok and Dudin, 2003].

To apply this algorithm, we should specify explicitly the one-step transition probability matrix of the Markov chain $\xi_n, n \geq 0$. Denote the matrices $P_{i,l}$ defined by the entries

$$P\{i_{n+1} = l, \nu_{n+1} = \nu' | i_n = i, \nu_n = \nu\}, \\ \nu, \nu' = \overline{0, W}.$$

To calculate the matrices $P_{i,l}$, which define probability of transitions of the Markov chain $\xi_n, n \geq 0$, between two successive service completion epochs, we should first calculate the matrices $P^{(j)}(n,t)$, which define the following conditional probabilities. The (ν, ν')-th entry of the matrix $P^{(j)}(n,t)$ is the probability to admit n customers during the time interval $(0,t]$ and to have the state ν' of the underlying process of the $BMAP$ ν_t at the epoch t conditional that the state of this process was ν at the epoch 0 and at most j customers can be admitted during the interval $(0,t]$, $n = \overline{0, j}$.

The matrices $P^{(j)}(n,t)$ are easily calculated in case of PA discipline:

$$P^{(j)}(n,t) = \left\{ \begin{array}{ll} P(n,t), & n < j, \\ \sum\limits_{l=j}^{\infty} P(l,t), & n = j, \end{array} \right. \quad (1)$$

where the matrices $P(n,t)$ are defined (see e. g. [Lucantoni, 1991]) as the coefficients in the following matrix expansion:

$$e^{D(z)t} = \sum_{n=0}^{\infty} P(n,t) z^n. \quad (2)$$

In case of CA and CR disciplines, we have no chance to present the matrices $P^{(j)}(n,t)$ in a simple form like (1), (2). It is the main reason for the lack of results in literature for these admission disciplines.

Despite the presence of the formulas (1), (2), the probability the matrices $P^{(j)}(n,t)$ are actually calculated not from expansion into series (2), but by means of Lucantoni's procedure based on the concept of uniformization. This procedure, see [Lucantoni, 1991], consists of the following. Let ψ be defined as $\psi = \max\limits_{\nu=\overline{0,W}} (-D_0)_{\nu,\nu}$. Then

$$P(n,t) = e^{-\psi t} \sum_{i=0}^{\infty} \frac{(\psi t)^i}{i!} U_n^{(i)}, \quad (3)$$

where the matrices $U_n^{(i)}$ are calculated recursively:

$$U_n^{(0)} = \left\{ \begin{array}{ll} I, & n = 0, \\ 0, & n > 0, \end{array} \right.$$
$$U_n^{(i+1)} = U_n^{(i)} (I + \psi^{-1} D_0) + \psi^{-1} \sum_{l=0}^{n-1} U_l^{(i)} D_{n-l},$$
$$i \geq 0,\ n \geq 0. \quad (4)$$

The formulas (3), (4) are obtained from the system of differential equations

$$\dot{P}(n,t) = \sum_{k=0}^{n} P(k,t) D_{n-k},\ n \geq 0, \quad (5)$$

which is easily derived from the difference equations for the matrices $P(n,t), n \geq 0$.

Following the same way for CA and CR disciplines, we can prove the statement.

Lemma 1 *The matrices $P^{(j)}(n,t)$ for the disciplines of CA and CR are calculated by*

$$P^{(j)}(n,t) = e^{-\psi t} \sum_{i=0}^{\infty} \frac{(\psi t)^i}{i!} U_n^{(i)}(j), \quad (6)$$

where the matrices $U_n^{(i)}(j)$ are calculated by the formulas

$$U_n^{(0)}(j) = \left\{ \begin{array}{ll} I, & n = 0, \\ 0, & n > 0, \end{array} \right. \quad (7)$$

$$U_n^{(i+1)}(j) = U_n^{(i)}(j) \left(I + \psi^{-1} (D_0 + \hat{D}_{j+1-n}) \right) +$$
$$+ \psi^{-1} \sum_{l=0}^{n-1} U_l^{(i)}(j) D_{n-l},$$
$$i \geq 0,\ j = \overline{0, N+1},\ n = \overline{0, j},$$
$$\hat{D}_l = \sum_{m=l}^{\infty} D_m \quad (8)$$

for CR discipline and by the formulas (7) and

$$U_n^{(i+1)}(j) = U_n^{(i)}(j)(I + \psi^{-1} D_0) +$$
$$+ \psi^{-1} \sum_{l=0}^{n-1} U_l^{(i)}(j) D_{n-l},$$
$$i \geq 0,\ j = \overline{0, N+1},\ n = \overline{0, j-1},$$
$$U_n^{(i+1)}(j) = U_n^{(i)}(j) \left(I + \psi^{-1} D(1) \right) +$$
$$+ \psi^{-1} \sum_{l=0}^{j-1} U_l^{(i)}(j) D_{n-l},$$
$$i \geq 0,\ j = \overline{0, N+1},\ n \geq j \quad (9)$$

for the case of the CA discipline.

The recursions (7), (8) and (7), (9) directly follow from the differential equations

$$\dot{P}^{(j)}(n,t) = P^{(j)}(n,t)(D_0 + \hat{D}_{j+1-n}) +$$
$$+ \sum_{l=0}^{n-1} P^{(j)}(l,t) D_{n-l},\ j = \overline{0, N+1},\ n = \overline{0, j} \quad (10)$$

for the case of CR discipline and the equations

$$\dot{P}^{(j)}(n,t) = P^{(j)}(n,t) D_0 + \sum_{l=0}^{n-1} P^{(j)}(l,t) D_{n-l},$$
$$j = \overline{0, N+1},\ n = \overline{0, j-1},$$
$$\dot{P}^{(j)}(n,t) = P^{(j)}(n,t) D(1) + \sum_{l=0}^{j-1} P^{(j)}(l,t) D_{n-l},$$
$$j = \overline{0, N+1},\ n \geq j \quad (11)$$

for the case of CA discipline.

The outline of derivation of the system (10) is the following.

As it was mentioned above, the entry $\left(P^{(j)}(n,t)\right)_{\nu,\nu'}$ of the matrix $P^{(j)}(n,t)$ is the probability to admit n customers at the interval $(0,t]$ and to have the state ν' of the underlying process $\nu_t, t \geq 0$ at the epoch t conditional that the state of this process was ν at the epoch 0 and at most j customers can be admitted, $\nu, \nu' = \overline{0, W}, n = \overline{0, j}, j = \overline{0, N+1}$.

Calculate the probability $\left(P^{(j)}(n, t+\Delta t)\right)_{\nu,\nu'}$ in terms of probabilities $\left(P^{(j)}(k,t)\right)_{r,r'}$ and probabilities of transitions of the underlying process $\nu_t, t \geq 0$, during the small interval $(t, t+\Delta t)$. As follows from the definition of the $BMAP$, the value $(D_k)_{r,r'} \Delta t + o(\Delta t)$ is the probability of the following event. The process ν_t makes a jump from the state r into the state r' during the interval $[t, t+\Delta t)$ with generation of a batch of size k.

Here $r, r' = \overline{0, W}$ for $k \geq 1$ and $r, r' = \overline{0, W}, r \neq r'$ for $k \geq 0$. Note, that according to CR discipline, the generated k-size batch is admitted to the system only if the sum of k and the number of previously admitted customers in the interval $(0, t]$ does not exceed the level j. The value $(1 + (D_0)_{r,r}\Delta t) + o(\Delta t)$ represents the probability to have no jumps of the process $\nu_t, t \geq 0$ from the state r during the time interval $[t, t + \Delta t)$.

Summarizing these considerations and using the formula of total probability, we derive the following difference equations:

$$\begin{gathered}\left(P^{(j)}(n, t + \Delta t)\right)_{\nu,\nu'} = \\ = \left(P^{(j)}(n, t)\right)_{\nu,\nu'} (1 + (D_0 + \hat{D}_{j+1-n})_{\nu',\nu'}\Delta t) + \\ + \sum_{\substack{r=0 \\ r\neq\nu'}}^{W} \left(P^{(j)}(n, t)\right)_{\nu,r} (D_0 + \hat{D}_{j+1-n})_{r,\nu'}\Delta t + \\ + \sum_{m=0}^{n-1}\sum_{r=0}^{W} \left(P^{(j)}(m, t)\right)_{\nu,r} (D_{n-m})_{r,\nu'}\Delta t + o(\Delta t), \\ n = \overline{0, j}, j = \overline{0, N+1}, \nu, \nu' = \overline{0, W}.\end{gathered}$$

By a routine way, these difference equations are reduced to the system of differential equations:

$$\begin{gathered}\left(\dot{P}^{(j)}(n, t)\right)_{\nu,\nu'} = \\ = \left(P^{(j)}(n, t)\right)_{\nu,\nu'} (D_0 + \hat{D}_{j+1-n})_{\nu',\nu'} + \\ + \sum_{\substack{r=0 \\ r\neq\nu'}}^{W} \left(P^{(j)}(n, t)\right)_{\nu,r} (D_0 + \hat{D}_{j+1-n})_{r,\nu'} + \\ + \sum_{m=0}^{n-1}\sum_{r=0}^{W} \left(P^{(j)}(m, t)\right)_{\nu,r} (D_{n-m})_{r,\nu'}, \\ n = \overline{0, j}, j = \overline{0, N+1}, \nu, \nu' = \overline{0, W}.\end{gathered}$$

Rewriting these equations in the matrix form, we get the system (10).

Derivation of the system (11) is implemented analogously.

Lemma 2 *The transition probability matrices $P_{i,l}$ are calculated by the formulas*

$$\begin{gathered}P_{0,l} = -(D_0 + \hat{D}_{N+2})^{-1} \sum_{k=1}^{l+1} D_k \times \\ \times \int_0^\infty P^{(N+1-k)}(l + 1 - k, t)dB(t),\ l = \overline{0, N},\end{gathered} \tag{12}$$

$$\begin{gathered}P_{i,l} = \int_0^\infty P^{(N+1-i)}(l + 1 - i, t)dB(t), \\ i = \overline{1, N},\ l = \overline{i-1, N}, \\ P_{i,l} = 0,\ i = \overline{i, N},\ l < i - 1,\end{gathered} \tag{13}$$

in case of CR discipline and by the formulas

$$\begin{gathered}P_{0,l} = (-D_0)^{-1} \sum_{k=1}^{l+1} D_k \times \\ \times \int_0^\infty P^{(N+1-k)}(l + 1 - k, t)dB(t),\ l = \overline{0, N}, \\ P_{0,l} = (-D_0)^{-1}\Big(D_{l+1}G + \\ + \sum_{k=1}^{N} D_k \int_0^\infty P^{(N+1-k)}(l + 1 - k, t)dB(t)\Big), \\ l > N,\end{gathered} \tag{14}$$

$$\begin{gathered}P_{i,l} = \int_0^\infty P^{(N+1-i)}(l + 1 - i, t)dB(t), \\ i = \overline{1, N},\ l \geq i - 1, \\ P_{i,l} = 0,\ i > N,\ l \neq i - 1 \text{ and } i > 0,\ l < i - 1, \\ P_{i,l} = G,\ i > N,\ l = i - 1,\end{gathered} \tag{15}$$

$$G = \int_0^\infty e^{D(1)t}dB(t) \tag{16}$$

in case of CA discipline.

The proof is quite clear if we take into account the probabilistic meaning of involved matrices. The entries of the matrix

$$\int_0^\infty P^{(N+1-k)}(l + 1 - k, t)dB(t)$$

define the probability of $l + 1 - k$ customers admittance into the system and corresponding transitions of the underlying process $\nu_t, t \geq 0$, of the $BMAP$ during the service time of one customer conditional that $N + 1 - k$ places were free at the service beginning epoch.

In case of CR discipline, the entries of the matrix

$$-(D_0 + \hat{D}_{N+2})^{-1}D_k = \int_0^\infty e^{(D_0+\hat{D}_{N+2})t}D_k dt$$

define the probability that the staying of the system in the idle state will be finished by arrival of a batch of size k and corresponding transitions of the underlying process $\nu_t, t \geq 0$, of the $BMAP$ take place during the idle interval, $k = \overline{1, N+1}$. Note, that the stability of the matrix $D_0 + \hat{D}_{N+2}$ is exploited in the presented relation.

Collection of formulas (6) — (9), (12) — (16) defines a simple algorithmic way for calculating the matrices $P_{i,l}$ for both customer's admission disciplines.

Finally, using the results presented in [Klimenok and Dudin, 2003], we can formulate the following statements.

Let $\pi(i,\nu) = \lim_{n\to\infty} P\{i_n = i, \nu_n = \nu\}$, $\nu = \overline{0,W}$ and $\boldsymbol{\pi}_i$ be the vectors of probabilities $\pi(i,\nu)$ listed in lexicographic order, $i \geq 0$.

Theorem 1 *In case of CR discipline, the probability vectors $\boldsymbol{\pi}_i$ are calculated by*

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_0\Phi_i,\ i = \overline{0,N}, \tag{17}$$

where the matrices Φ_i are calculated by

$$\Phi_0 = I, \\ \Phi_l = \sum_{i=0}^{l-1} \Phi_i \bar{P}_{i,l}(I - \bar{P}_{l,l})^{-1},\ l = \overline{1,N}. \tag{18}$$

The vector $\boldsymbol{\pi}_0$ is the unique solution of the system

$$\boldsymbol{\pi}_0(I - \bar{P}_{0,0}) = \mathbf{0}, \\ \boldsymbol{\pi}_0 \sum_{l=0}^{N} \Phi_l \mathbf{e} = 1. \tag{19}$$

The matrices $\bar{P}_{i,l}$ are calculated recurrently:

$$\bar{P}_{i,l} = P_{i,l} + \bar{P}_{i,l+1}G_l,\ i = \overline{0,N},\ l = \overline{i,N}, \\ \bar{P}_{i,N+1} = 0. \tag{20}$$

The matrices G_i are calculated by

$$G_{N-1} = (I - P_{N,N})^{-1}P_{N,N-1}, \\ G_i = (I - \sum_{l=i+1}^{N} P_{i+1,l}G_{l-1}G_{l-2}\cdot\ldots\cdot G_{i+1})^{-1}P_{i+1,i}, \\ i = \overline{0,N-2}. \tag{21}$$

Theorem 2 *In case of CA discipline, the stationary distribution vectors $\boldsymbol{\pi}_i$ are calculated by*

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_0\Phi_i,\ i \geq 0. \tag{22}$$

The matrices Φ_i and the vector $\boldsymbol{\pi}_0$ are defined by the formulas (18) — (21), where N is replaced with infinity and the matrices G_i for $i \geq N$ are equal to the matrix G, which is defined by the formula (16).

Remark. Formulas (17) — (22) define stable procedures for calculation of the steady state distribution $\boldsymbol{\pi}_i, i \geq 0$, because all matrices involved into recursive formulas are non-negative. Note that matrices of the form $(I - H)^{-1}$, which appear in (18) and (21), are non-negative because the matrices standing for H in (18) and (21) are irreducible sub-stochastic.

THE ARBITRARY TIME DISTRIBUTION AND THE LOSS PROBABILITY

Now we are able to compute the arbitrary time stationary probabilities $p(i,\nu) = \lim_{t\to\infty} P\{i_t = i, \nu_t = \nu\}$, $\nu = \overline{0,W}$. Let $\mathbf{p}_i$, $i \geq 0$ be the vectors of these probabilities listed in lexicographic order.

Theorem 3 *In case of CR discipline, the vectors $\mathbf{p}_i$, $i = \overline{0,N+1}$ are calculated by*

$$\mathbf{p}_0 = \tau^{-1}\boldsymbol{\pi}_0(-1)(D_0 + \hat{D}_{N+2})^{-1}, \tag{23}$$

$$\mathbf{p}_i = \tau^{-1}\Big(\boldsymbol{\pi}_0(-1)(D_0 + \hat{D}_{N+2})^{-1}\times \\ \times \sum_{k=1}^{i}\int_0^\infty D_k P^{(N+1-k)}(i-k,t)(1-B(t))dt + \\ + \sum_{k=1}^{\min\{i,N\}} \boldsymbol{\pi}_k \int_0^\infty P^{(N+1-k)}(i-k,t)(1-B(t))dt\Big), \\ i = \overline{1,N+1}, \tag{24}$$

where the average inter-departure time τ is calculated by

$$\tau = b_1 + \boldsymbol{\pi}_0(-1)(D_0 + \hat{D}_{N+2})^{-1}\mathbf{e}. \tag{25}$$

Theorem 4 *In case of CA discipline, the vectors $\mathbf{p}_i$, $i \geq 0$ are calculated by*

$$\mathbf{p}_0 = \tau^{-1}\boldsymbol{\pi}_0(-D_0)^{-1}, \tag{26}$$

$$\mathbf{p}_i = \tau^{-1}\Big(\boldsymbol{\pi}_0(-D_0)^{-1}\times \\ \times \sum_{k=1}^{i}\int_0^\infty D_k P^{(N+1-k)}(i-k,t)(1-B(t))dt + \\ + \sum_{k=1}^{i} \boldsymbol{\pi}_k \int_0^\infty P^{(N+1-k)}(i-k,t)(1-B(t))dt\Big), \\ i = \overline{1,N+1}, \tag{27}$$

$$\mathbf{p}_i = \tau^{-1}\Big(\boldsymbol{\pi}_0(-D_0)^{-1}\times \\ \times \sum_{k=1}^{N+1}\int_0^\infty D_k P^{(N+1-k)}(i-k,t)(1-B(t))dt + \\ + \sum_{k=1}^{N} \boldsymbol{\pi}_k \int_0^\infty P^{(N+1-k)}(i-k,t)(1-B(t))dt + \\ + \big(\boldsymbol{\pi}_0(-D_0)^{-1}D_i + \boldsymbol{\pi}_i\big)\int_0^\infty e^{D(1)t}(1-B(t))dt\Big), \\ i > N+1, \tag{28}$$

where the average inter-departure time τ is calculated by

$$\tau = b_1 + \boldsymbol{\pi}_0(-D_0)^{-1}\mathbf{e}. \tag{29}$$

Theorems 3 and 4 are proven by means of the theory of Markov renewal processes, see [Cinlar, 1975]. Here the entries of the matrix

$$\int_0^\infty P^{(N+1-k)}(l+1-k,t)(1-B(t))dt$$

define the probability of $l+1-k$ customers admittance into the system and corresponding transitions of the underlying process ν_t, $t \geq 0$, of the $BMAP$ during the time interval between an arbitrary time epoch, when the server is busy, and the last previous service beginning epoch conditional that $N+1-k$ places were free at the service beginning epoch.

Having the stationary state distribution of the system been computed, we can easy calculate different performance measures of the system and effectively plan the capacity of the system. Probability of an arbitrary customer loss in one of the most important performance measures of the system under study. The following two theorems provide the ways to calculate this probability.

Theorem 5 *The probability P_{loss} of an arbitrary customer loss is calculated by*

$$P_{loss} = 1 - \lambda^{-1} \sum_{i=0}^{N} \sum_{k=1}^{N+1-i} k\mathbf{p}_i D_k \mathbf{e} \qquad (30)$$

in case of CR discipline and by

$$P_{loss} = 1 - \lambda^{-1} \sum_{i=0}^{N} \sum_{k=1}^{\infty} k\mathbf{p}_i D_k \mathbf{e} \qquad (31)$$

in case of CA discipline.

Theorem 6 *The probability P_{loss} of an arbitrary customer loss is calculated by*

$$P_{loss} = 1 - (\tau\lambda)^{-1} \qquad (32)$$

in case of PA, CR, CA disciplines.

Theorems 5 and 6 can be proven following to [Klimenok, 2002].

NUMERICAL RESULTS

Presented analytical results provide the background for creating stable numerical procedures for calculating the stationary distributions at embedded and arbitrary epochs and the loss probability in particular.

This section contains some results of calculations. The results demonstrate the feasibility of the algorithms proposed in this paper and describe the performance of the queueing model under study. In particular, they illustrate the dependence of the loss probability on: (i) the discipline of customers admission; (ii) correlation in the input flow; (iii) variation in the input flow; (iv) variation of the service time; (v) finite buffer capacity; (vi) load $\rho = \lambda b_1$.

Consider three different $BMAP$s having the same fundamental rate $\lambda = 12.8153$ and the variation coefficient c_{var} equal to two. All these $BMAP$s are defined by the matrices D_0 and D_k, $k = \overline{1,5}$. These matrices are defined by $D_k = Dq^{k-1}(1-q)/(1-q)^5$, $k = \overline{1,5}$, where $q = 0.8$ and D is a fixed matrix.

The $BMAP$ input coded as $BMAP_1$ is defined by the matrices

$$D_0 = \begin{pmatrix} -13.334625 & 0.588578 & 0.617293 \\ 0.692663 & -2.446573 & 0.422942 \\ 0.682252 & 0.414363 & -1.635426 \end{pmatrix},$$

$$D = \begin{pmatrix} 11.546944 & 0.363141 & 0.218669 \\ 0.384249 & 0.865869 & 0.080851 \\ 0.285172 & 0.04255 & 0.211089 \end{pmatrix}.$$

It has the correlation coefficient of successive inter-arrival times equal to 0.1.

The next $BMAP$ coded as $BMAP_2$ and having the correlation coefficient 0.2 is defined by the matrices

$$D_0 = \begin{pmatrix} -15.732675 & 0.606178 & 0.592394 \\ 0.517817 & -2.289674 & 0.467885 \\ 0.597058 & 0.565264 & -1.959665 \end{pmatrix},$$

$$D = \begin{pmatrix} 14.1502 & 0.302098 & 0.081805 \\ 0.107066 & 1.03228 & 0.164627 \\ 0.08583 & 0.197946 & 0.513566 \end{pmatrix}.$$

The third $BMAP$ coded as $BMAP_3$ and having the correlation coefficient 0.3 is defined by the matrices

$$D_0 = \begin{pmatrix} -25.53984 & 0.393329 & 0.361199 \\ 0.14515 & -2.2322 & 0.200007 \\ 0.295961 & 0.3874445 & -1.752618 \end{pmatrix},$$

$$D = \begin{pmatrix} 24.24212 & 0.466868 & 0.076323 \\ 0.034097 & 1.666864 & 0.186082 \\ 0.009046 & 0.255481 & 0.804685 \end{pmatrix}.$$

The fourth input process, which is coded as M^X, is a stationary group Poisson process having the fundamental rate $\lambda = 12.8153$ and the same batch size distribution as the $BMAP_1$-$BMAP_3$. The correlation coefficient of the successive inter-arrival times is zero for the M^X.

Figure 1 illustrates the form of dependence of the loss probability on the buffer capacity N and the admission discipline for $BMAP_2$ and Erlangian service time distribution $B(t) = \mu \int_0^t \frac{(\mu x)^{k-1}}{(k-1)!} e^{-\mu x} dx$, $\mu = 28$, $k = 2$, $t \geq 0$. The load ρ of the system is equal to 0.915381. As it is intuitively expected, CR discipline provides the maximal value of P_{loss}, while CA provides the minimal one.

Figures 2 — 4 illustrate the dependence of P_{loss} on the buffer capacity and correlation in the $BMAP$ for different admission disciplines.

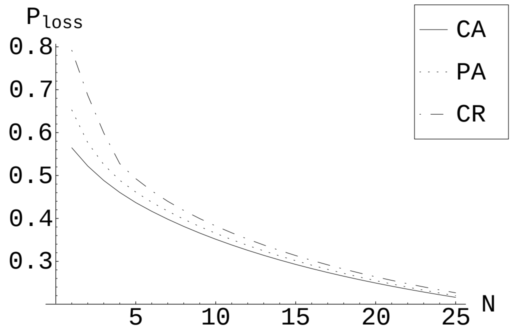

Figure 1: The Loss Probability P_{loss} for CA, PA, CR Disciplines

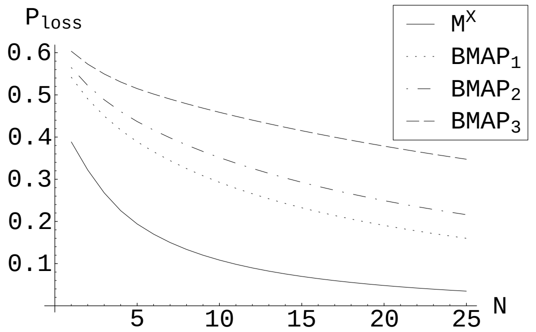

Figure 2: The Loss Probability P_{loss} for CA Discipline

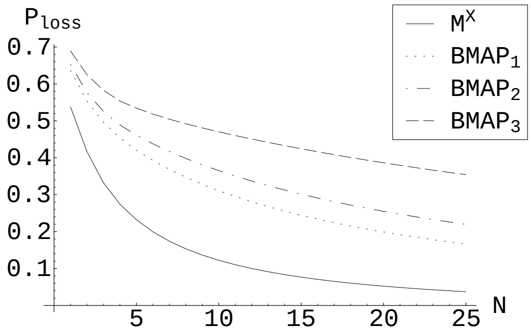

Figure 3: The Loss Probability P_{loss} for PA Discipline

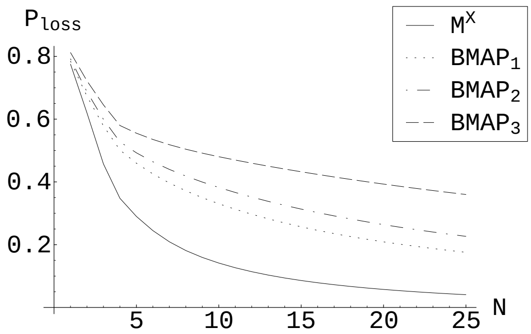

Figure 4: The Loss Probability P_{loss} for CR Discipline

Important conclusion follows from the figures 2-4. Correlation in the input flow essentially affects on the value of the loss probability.

Now consider four $BMAP$s having the squared variation coefficient equal to $c_{var}^2 = L, L = 4, 6, 22, 54$, denoted as $BMAP^L$. The fundamental rate λ of these $BMAP$s is equal to 5, correlation coefficient is 0.3. All these $BMAP$s are defined by the matrices D_0 and D_k, $k = \overline{1,5}$. As above, these matrices are defined by $D_k = Dq^{k-1}(1-q)/(1-q)^5$, $k = \overline{1,5}$, where $q = 0.8$ and D is a fixed matrix.

The $BMAP^4$ is the same as $BMAP_3$ defined above. The $BMAP^6$ is defined by the matrices

$$D_0 = \begin{pmatrix} -16.0013 & 0.201548 & 0.187506 \\ 0.167115 & -1.577802 & 0.161753 \\ 0.196437 & 0.218746 & -1.018238 \end{pmatrix},$$

$$D = \begin{pmatrix} 15.353199 & 0.250430 & 0.008616 \\ 0.1022775 & 1.032086 & 0.114072 \\ 0.017409 & 0.111128 & 0.474517 \end{pmatrix}.$$

The $BMAP^{22}$ is given by the matrices

$$D_0 = \begin{pmatrix} -16.196754 & 0.090698 & 0.090698 \\ 0.090698 & -0.545155 & 0.090698 \\ 0.090698 & 0.090698 & -0.313671 \end{pmatrix},$$

$$D = \begin{pmatrix} 15.949221 & 0.066138 & 0 \\ 0.033069 & 0.297622 & 0.033069 \\ 0 & 0.013228 & 0.119049 \end{pmatrix}.$$

The $BMAP^{54}$ is defined by the matrices

$$D_0 = \begin{pmatrix} -16.268103 & 0.040591 & 0.040591 \\ 0.040591 & -0.223594 & 0.040591 \\ 0.040591 & 0.040591 & -0.132968 \end{pmatrix},$$

$$D = \begin{pmatrix} 16.161048 & 0.25893 & 0 \\ 0.012947 & 0.116519 & 0.012947 \\ 0 & 0.005179 & 0.046608 \end{pmatrix}.$$

Figures 5 — 7 convince that the variation of inter-arrival times also essentially affects on the value of the loss probability. The general conclusion from the results of the presented experiments is the following. Design of finite buffer systems, which does not take into account correlation and variation of inter-arrival times, e. g., by assuming the stationary Poisson input flow or even the

PH type arrival process, can lead to essential errors. Prediction of the loss probability based on the model with the Poisson input flow is too optimistic. More extensive numerical experiments show that the error in estimation of the loss probability can be of 5–6 orders.

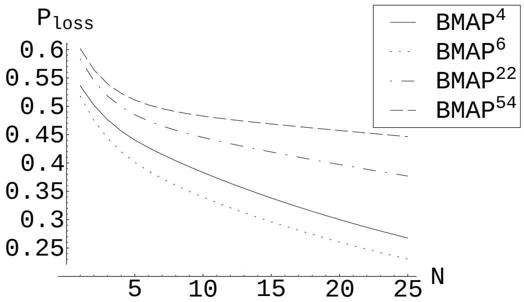

Figure 5: The Loss Probability P_{loss} for CA Discipline

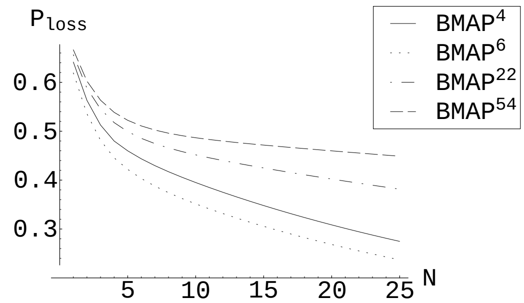

Figure 6: The Loss Probability P_{loss} for PA Discipline

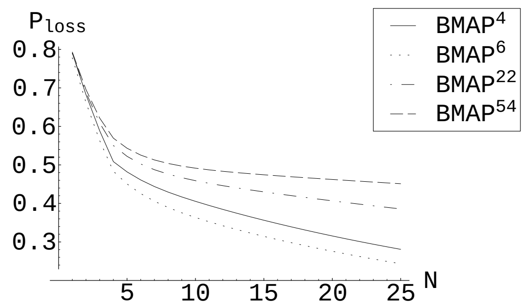

Figure 7: The Loss Probability P_{loss} for CR Discipline

Now consider the question whether the value of the loss probability is sensitive with respect to the service time distribution. To this end, we take four different service time distributions having the same expectation $b_1 = 1/32$ but different variation:

- degenerate distribution

$$B_1(t) = \begin{cases} 0, & t < 1/32, \\ 1, & t \geq 1/32; \end{cases}$$

- exponential distribution

$$B_2(t) = 1 - e^{-32t},\ t \geq 0;$$

- hyper-exponential distribution

$$B_3(t) = 1 - 1/2\left(e^{-272t} + e^{-17t}\right),\ t \geq 0;$$

- lognormal distribution

$$B_4(t) = \int_0^t \frac{1}{\sqrt{2\pi}\beta x} e^{-\frac{(\ln x - \alpha)^2}{2\beta^2}} dx,$$
$$\alpha = -5.46574,\ \beta = 2,\ t \geq 0.$$

The variation coefficients of these distributions are equal to 0, 1, 1.6, 7.3 correspondingly.

Figures 8 — 10 illustrate the dependence of the loss probability on service time variation and the buffer size N. Here the input flow is $BMAP_2$ and the service time distributions are $B_k(t)$, $k = \overline{1,4}$. The load of the system is 0.400479. The curve coded as B_k corresponds to the service time distribution function $B_k(t)$. The figures show that service time variation also impacts on the value of the loss probability P_{loss}.

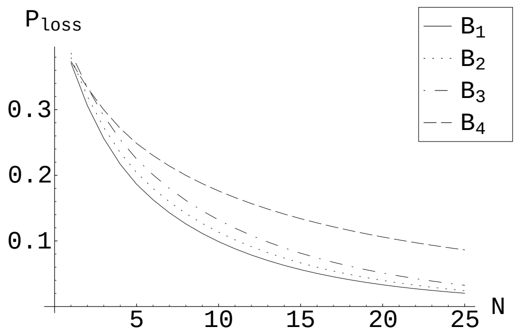

Figure 8: The Loss Probability P_{loss} for CA Discipline

In all previous experiments, we presented dependence of loss probability on the buffer capacity N. Figure 11 illustrates the dependence of the loss probability on the load of the system ρ and the admission discipline for $BMAP_2$, buffer capacity $N = 3$ and degenerate service time distribution.

Figures 12 — 14 are presented to better visualize the dependence of the loss probability on the buffer capacity N and the load ρ of the system for $BMAP_2$, degenerate service time distribution and different admission strategies. One can see

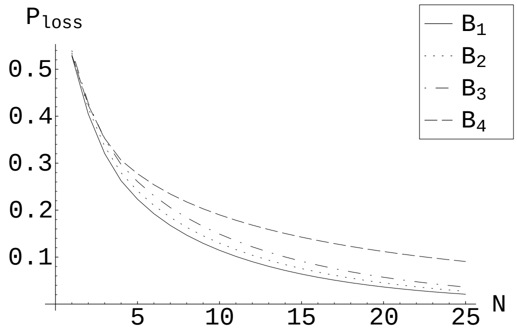

Figure 9: The Loss Probability P_{loss} for PA Discipline

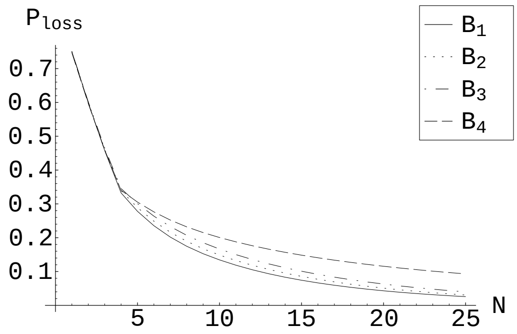

Figure 10: The Loss Probability P_{loss} for CR Discipline

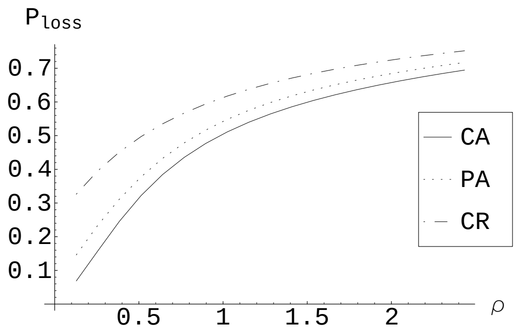

Figure 11: The Loss Probability P_{loss} for CA, PA, CR Disciplines

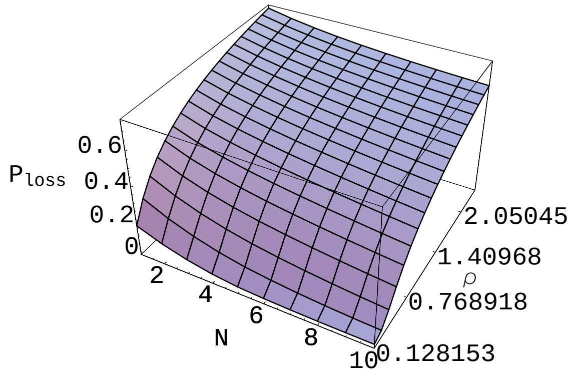

Figure 12: The Loss Probability P_{loss} for CA Discipline

that in the case of very heavy traffic, $\rho > 2$, the difference between the admission disciplines PA and CR is not very essential. Discipline CA provides better performance of the system. Dependence on the admission discipline is very essential in case of small buffer capacity. Deformation of surfaces on figures 12 — 14 for relatively moderate load of the system around the line $N = 5$ is explained by the fact that 5 is the maximal size of a batch in the considered $BMAP$s.

CONCLUSION

The $BMAP/G/1/N$ system with complete admission and complete rejection disciplines, which are popular in real life systems, is investigated while only partial admission discipline was considered in [Dudin et al., 2002a].

The queue length distribution at service completion and arbitrary time is calculated and formulas for the loss probability are given. The influence of: correlation and variation in the input; variation of the service time; admission discipline; buffer capacity; system load on the value of loss probability is illustrated by numerical examples.

The presented results evidently show that behavior of the input flow very essentially effects the value of the loss probability. So, the problem of the modern finite systems design can not be satisfactory solved based on the models of the $M/M/1/N$ or $M/G/1/N$ type which are common in engineering literature. Results of our paper allow to solve this problem effectively. Presented algorithms are realized as computationally stable computer programs.

Corresponding program modules are included into software packages “SIRIUS++” and ”SIRIUS-C”, see, e. g. [Dudin et al., 2002b], [Dudin et al., 2004], created for calculation of characteristics of the different queueing systems with the $BMAP$ input. This software has friendly interface and is available from the authors.

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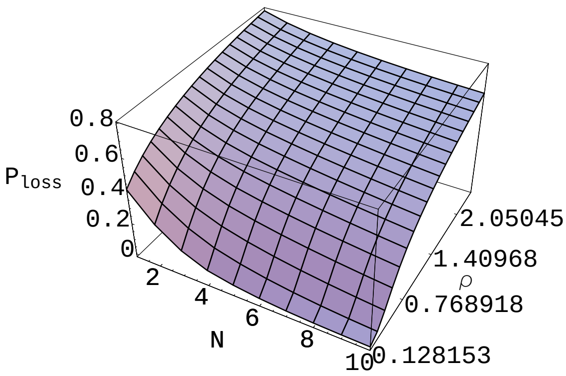

Figure 13: The Loss Probability P_{loss} for PA Discipline

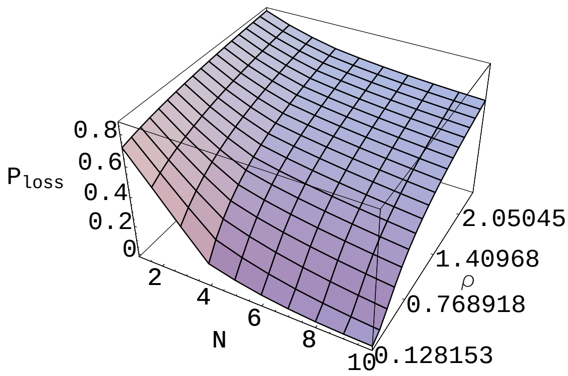

Figure 14: The Loss Probability P_{loss} for CR Discipline

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