# A New Approach to Searching Lyapunov Function Candidates for Autonomous Nonlinear Systems

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*Abstract* — **Stability of nonlinear systems is a problem of fundamental importance in system engineering. Specifically, the computation of a Lyapunov Function presents one of the tools enabling the study of the stability of nonlinear systems. The aim of this work is to study the Lyapunov approaches for polynomial systems. These approaches have been investigated in order to develop numerical algorithms based on the synthesis of Polynomial Lyapunov Functions. We proceed in two steps: Firstly, we implement a Threshold Accepting Algorithm technique to determine a candidate Lyapunov function. Secondly, we use an optimization strategy based on a Linear Matrix Inequality (LMI) to compute the Region of Attraction (RA). The parameters of the Lyapunov Function are computed by combining Threshold Accepting Algorithms (TAA) and LMI. The proposed approach yields a larger stability region for polynomial systems than an existing method does. Examples are given to illustrate the efficiency of the proposed approach.**

***Keywords - Stability, Polynomial Lyapunov Function, Threshold Accepting Algorithm, LMI.***

## I. INTRODUCTION

The Lyapunov stability theory proves to be a handy general theory dedicated to the study of the stability of nonlinear system equilibrium point [20]. In fact, this theory was first presented by Alexander M. Lyapunov in 1892 and it entails two methods: Lyapunov's indirect and direct methods. The first method, which is called Lyapunov's first Method ([1], [6], [19], [20]), required the linearization of a particular nonlinear system around an operating point. Indeed, this method is worth being pointed out since it ultimately provides a localized stability analysis technique that endows us with qualitative information about the stability uniquely in relation to the operating point in question. In spite of being a blatant feature of the system, it does in no way bestow any information with respect to the global stability of the nonlinear system because, for the general nonlinear system, the instability of one or more of its equilibrium states does not signify a global instability ([19], [20]). Lyapunov's second method, which is usually named Lyapunov's Second or Direct Method, has proven to be a more general and powerful approach as it allows the potential global stability of the general nonlinear system to be explored. For this reason, it does not undergo the negative aspects shown in Lyapunov's first method [21]. Essentially, this approach requires the construction of a Lyapunov Function; a concept inspired by the instinctive knowledge that the energy in the proximity of the equilibrium state of a physical system is constantly reduced. This means that the equilibrium is stable.

A Lyapunov Function is simply an expression of this energy concept, where the analysis of a specific nonlinear system is reduced to the exploration of the features of its corresponding Lyapunov Function. Accordingly, the main advantage of this method is that it does not need an analytical or numerical answer ([10], [11], [23], [25]). Thus, it enjoys a more functional power. However, there is a problem, as the requisite Lyapunov Function, together with its time derivative, is likely to meet strict restraints. Furthermore, there are still no methodologies to obtain such a function. Additionally, the failure to obtain a specific Lyapunov Function for a certain nonlinear system does not suppose that the equilibrium point or, possibly, the global nonlinear system under examination is unstable. Hence, this means that this technique is adequate but not essential for stability. To surmount this hurdle, various tactics have been proposed in the literature ([2], [3], [4], [5], [7], [8], [9], [22], [23], [24]).

In order to find a general tool for the creation of the Lyapunov Function, we take advantage of the Threshold Accepting Algorithm (TAA) to calculate the Lyapunov Function contender. These algorithms are variations of the simulated annealing [20]: the difference emerges in the acknowledgement of degradation at each stage. In the Simulated Annealing, the conclusion is attained in accordance with the Metropolis criterion. These algorithms have been used to resolve optimization difficulties with interesting conclusions. The major complexity of this approach resides in the thresholds for a specific application.

A Region of Attraction (RA) is the set of initial conditions for which state trajectories converge to an equilibrium point

([12], [13]). Therefore, it is important to be able to specify the shape of this region. For this purpose, we can use a Lyapunov Function (see for instance [15], [16], [17]). Indeed, for a given Lyapunov Function providing local stability of the equilibrium point, the largest estimated region of attraction, whose shape is fixed by the Lyapunov Function itself, is defined as the largest level set of the Lyapunov Function included in the region where its derivative is negative.

The Lyapunov Function describing the shape of the largest estimated RA is assumed to be polynomial and is chosen arbitrarily. In [14], the author proposes to exploit relaxations based on sum of squares polynomials in order to prove that the lower bound of the maximum achievable largest estimated RA can be computed via a generalized eigenvalue problem. The advantage is that the problem is formulated as a quasi-convex LMI optimization. The purpose of this paper is to propose an exact method for the enlargement and the maximization of the RA. The starting point of this work is the method developed in [14]. The objective is to improve the results by combining a TAA as an advanced optimization routine to LMI techniques in order to maximize the RA.

In order to get an explicit expression of the estimated RA, we investigate a Lyapunov approach based on a parameterized Lyapunov Function. Since the RA is related to the Lyapunov Function, the idea consists in choosing the best parameters in order to obtain the largest RA. These parameters are computed as solutions of an optimization problem. We propose to apply a TAA to solve this problem. This yields an optimal set of parameters of Lyapunov Function, which is used to solve the LMI optimization problem derived from [14].

The RA is the set of states that can be steered towards the terminal region. The size of the RA depends on the chosen Lyapunov function. This leads to a greater number of parameters and therefore, to a greater computational effort. In this work, the optimization problem formulation, and hence the computational effort, is similar to the original one but with a larger domain using TAA. The main advantages of the TAA are their robustness and efficiency in different environments covering various applications.

The paper at hand is organized as follows: In Section 2, we describe the class of systems considered and recall some basic notions. In Section 3, the stability analysis is studied. The proposed agreed algorithm is described in Section 4. The calculation results of two examples are set out in Section 5. Finally, our paper is ended up with a concluding paragraph in Section 6.

## II. Problem Formulation

### A. *Notation and Polynomial representation*

The notation adopted in the paper is as follows: $0_n$ origin of $R_n$ ; $I_n$ identity matrix $n \times n$ ; $A^T$ : transpose of matrix $A$ ; $A > 0 (A \geq 0)$ : symmetric positive definite (semi definite) matrix $A$ ; $A \otimes B$ :Kronecker's product of matrices $A$ and $B$ ; $x^{\{\delta\}} \in R^{\varsigma(n,\delta)}$ : vector containing all monomials of degree less or equal to $\delta$ in $x$ but the constant term. For example $x^{\{\delta\}} = \left[ x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^{\delta} \right]'$ . Quantities $\varsigma(n,\delta)$ is given by

$$\varsigma(n,\delta) = \frac{(n+\delta)!}{n!\delta!} - 1 \qquad (1)$$

### B. *Problem formulation*

Consider the continuous-time polynomial system

$$\dot{x} = f(x) \qquad (2)$$

where $f$ is polynomial function such that $f(0) = 0$ . In the following, $x \in R^n$ is the state vector. The equilibrium point of interest is the origin. Before proceeding further, we will give some preliminary results.

Let us consider $V(x) \in R$ , a positive definite, radially unbounded and continuously differentiable function. The bounded set

$$\Omega(c) = \left\{ x \in R^n / V(x) \leq c \right\} \qquad (3)$$

is an estimate of the region of attraction (RA) if $\Omega(c) \subset D$ where

$$D = \left\{ x \in R^n / \dot{V}(x) < 0 \right\} \cup \{0\} \qquad (4)$$

The time derivative of $\dot{V}(x)$ along the trajectory of system (2) is given by:

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V(x)}{\partial x} f(x) \\ &= L_f V(x) \end{aligned} \qquad (5)$$

$L_f V(x)$ is the Lie derivative of $V(x)$ along the polynomial vector $f(x)$ . The largest estimate of the RA is given by $\Omega(c^*)$

where

$$\begin{aligned} c^* &= \inf_{x \in R^n} V(x) \\ \textit{such that} \quad & \dot{V}(x) = 0 \end{aligned} \qquad (6)$$

The optimal value of $\hat{c}^*$ is obtained by

$$\hat{c}^* = \sup \ c^* \qquad (7)$$

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It has been proved in [14] that for any given $c \in R, c \le c^*$ if there exist $s(x)$ a positive definite polynomial such that

$$\dot{V}(x)+(c-V(x))s(x)<0 \tag{8}$$

The polynomial degrees of $V(x)$ and $\dot{V}(x)$ are $2\delta_V$ and $\delta_L$ respectively. We choose $s(x)$ degree to be $2\delta_s$ such that

$$\delta_s \ge \frac{\delta_L}{2}-\delta_V \tag{9}$$

It follows that the degree of the polynomial

$$t(x,c,s(x))=\dot{V}(x)+(c-V(x))s(x) \tag{10}$$

is equal to $2\delta_m$ where $\delta_m = \delta_s + \delta_V$ .

A Square Matricial Representation (SMR) and Complete Square Matrix Representation (CSMR) of polynomials (see for example [14]) are used in order to get an appropriate optimization problem. The CSMR provides all the possible representations of a polynomial in terms of a quadratic form.

The CSMR matrix of $t(x,c,s(x))$ is given by

$$T(\alpha,c,S)=D_f(\alpha)+cW_1(S)-W_2(S) \tag{11}$$

where $D_f(\alpha)$ is the CSMR of $L_f V(x)$, is the SMR of $L_{(g,U)}V(x), W_1(S)$ and $W_2(S)$ are the SMR of $s(x)$ and $V(x)s(x)$.

The condition (7) with (11) implies that if

$$\hat{c}^* = \sup_{\alpha,S>0} c \quad such\ that \quad T(\alpha,c,S)<0 \tag{12}$$

Then $\hat{c}^* \le c^*$ .

This leads to a non-convex problem (since c multiplies the parameters of S in $T(\alpha,c,S)$ . The following Theorem is a reformulation of this problem as a generalized eigenvalue problem (GEVP), which enables us to overcome this limitation.

***Theorem*** ([14])

The lower bound $\hat{c}^*$ is given by

$$\hat{c}^* = \frac{-\lambda^*}{1+\mu\lambda^*} \tag{13}$$

with $\lambda^*$ solution of the following GEVP

$$\lambda^* = \inf_{\alpha,S>0,\lambda} \lambda$$
$$such\ that \begin{cases} 1+\mu\lambda>0 \\ S>0 \\ \lambda W(S) > D_f(\alpha)-W_2(S) \end{cases} \tag{14}$$

where $\mu$ can be any positive scalar and

$$W(s) = K'\left(\begin{bmatrix} 1 & 0 \\ 0 & \mu V \end{bmatrix} \otimes S\right)K \tag{15}$$

where $\otimes$ represents Kronecker's product and the matrix $k$ satisfies.

$$\begin{bmatrix} 1 \\ x^{\{\delta_v\}} \end{bmatrix} \otimes x^{\{\delta_s\}} = Kx^{\{\delta_m\}} \tag{16}$$

where

$x^{\{\delta_m\}} \in R^{\varsigma(n,\delta_m)}, \alpha \in R^{\tau(n,\delta_m)}, x^{\{\delta_V\}} \in R^{\varsigma(n,\delta_V)}, x^{\{\delta_s\}} \in R^{\varsigma(n,\delta_s)}$ and $K \in R^{\varsigma(n,\delta_s)(\varsigma(n,\delta_V)+1)\times\varsigma(n,\delta_m)}$

The quantities $\varsigma(n,\delta_m)$ and $\tau(n,\delta_m)$ are given by

$$\varsigma(n,\delta_m)=\frac{(n+\delta_m)!}{n!\delta_m!}-1 \tag{17}$$

$$\tau(n,\delta_m)=\frac{1}{2}\varsigma(n,\delta_m)\left(\varsigma(n,\delta_m)+1\right)-\varsigma(n,2\delta_m)+n \tag{18}$$

## III. THRESHOLD ACCEPTING ALGORITHMS FOR NONLINEAR STABILITY ANALYSIS

The problem investigated in this paper is the computation of the domain of attraction. The considered LFs are polynomials in $x$ with coefficient polynomials depending on $p_i,(i=1,2,3)$ and the matrix $A$ . $A$ is the linearization around the origin of the system (2)which is supposed to be locally asymptotically stable:

$$A=\left.\frac{\partial f(x)}{\partial x}\right|_{x=0} \tag{19}$$

### A. *Lyapunov Theory*

This section deals with a general Lyapunov theory for the stability of a linear system given, a linear dynamical system

$$\dot{x}=Ax \tag{20}$$

is asymptotically stable if each positive definite matrix $Q$ $\exists$ a positive definite matrix $P$ such that Lyapunov's equation

$$A^T P+PA=-Q \tag{21}$$

is satisfied.

In the case a Lyapunov function is given by

$$V = x^T Px \quad (22)$$

The function $\dot{V}(x)$ is locally negative definite if and only if $PA + A^T P < 0$, where $P$ is the definite the quadratic part $x^T Px$ of $V(x)$. The system (2) admits a Lyapunov Function if the positive definite matrices $P$ satisfying this condition can be parameterized through the Lyapunov equation
$P = P^T > 0$ such that

$$PA + A^T P = Q < 0 \quad (23)$$

where $Q$ is any negative definite matrix.

As reviewed earlier, the stability of any linearization systems can be concluded via the quadratic Lyapunov function. Regarding this, the matrices $P$ and $Q$ must exist satisfy the Lyapunov equation. Generic Threshold Accepting Algorithms have been modified to search for these matrices.

The purpose of this section is to present a method to estimate analytically the RAS. We consider for this a quadratic Lyapunov function of the form:

$$V(x) = x^T Px \quad (24)$$

where $P = P^T > 0$

In order to simplify the presentation, we suppose that $P$ is two-dimensional. The results are generalized to the case of a larger size matrix. We suppose that

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \quad (25)$$

Then

$$V(x) = p_1 x_1^2 + 2p_2 x_1 x_2 + p_3 x_2^2 \quad (26)$$

Using such a Lyapunov function we can express the Domain of Attraction in terms of ellipsoid in the $(x_1, x_2)$ plane. The idea of the following consists in estimating the parameters $p_i, (i = 1,2,3)$ via a Threshold Accepting Algorithms. By combining this algorithm with an LMI optimization we can obtain the largest ellipsoid.

The candidate solutions $p_i, (i = 1,2,3)$ are chosen arbitrarily with the Threshold Accepting Algorithms.

The Threshold Accepting Algorithms (TAA) method is a variant of the classical simulated annealing algorithm, originally introduced by Dueck and Scheuer [18]. In this respect, TAA abridged the simulated annealing procedure by excluding the element of probability in accepting inferior solutions. What is more, TAA introduced a deterministic threshold and an inferior solution is accepted if its disparity with the existing solution is smaller than or equal to the threshold. The major components of TAA are the functions that decide the lowering of the threshold during the course of the procedure, preventing criteria as well as the methods used to generate primary and neighboring solutions. The main benefits of TAA are its theoretic simplicity and its exceptional performance on various combinatorial optimization problems. On the assumption that $\mathrm{P}$ is the set of all feasible solutions of the problem, TAA starts with a primary solution $P \in \mathrm{P}$ which may be generated randomly or used as a simple method. The method proceeds in an iterative manner. In each iteration, the algorithm determines if the new solution $P'$ is less than the current solution $P_c$ then the original one will be replaced by the new one; otherwise, another solution will be generated. If the current solution is less than the best solution so far $P_b$, then the best solution will be replaced by $P_c$.

Considering the nonlinear system (2), the procedural list below exhibits the Threshold Accepting Algorithm steps.

TABLE 1. SIMPLE THRESHOLD ACCEPTING ALGORITHMS

| | |
|---|---|
| **Step0:** | Initialized solutions $P$ and $Q$ by zeroing all elements. |
| **Step1:** | Generate randomly the elements $p_i, (i = 1,2,3)$ of *P*. Determine positive definiteness of *P*. If $P$ is not positive definite, go to **step1**. |
| **Step 2:** | Compute $Q$ based on $P$ in the Lyapunov equation $PA + A^T P = Q < 0$. Determine the negative definiteness of *Q*. If $Q$ is not negative definite, go to **step 1**. |
| **Step 3:** | Accept solutions $P$ and $Q$. Exist. |

## IV. MAIN ALGORITHM FORMULATION

This section is dedicated to the formulation of the final flowchart which leads to the RA estimation of the studied system. The obtained solution is specified by the definition of a maximal quadratic Lyapunov function.

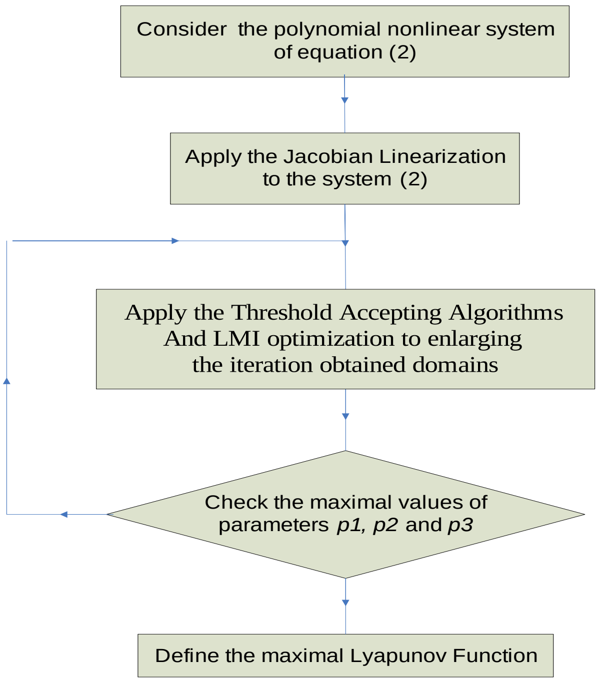


Figure 1. Flowchart of the advanced LMI Optimization Algorithm for Maximizing the Region of Attraction.

## V. EXAMPLES

### A. *Example1*

Consider the following nonlinear system

$$\dot{x}_1 = -2x_1 - x_1x_2$$
$$x_2 = x_1x_2 - x_2$$

According to the algorithm, proposed in paragraph IV, we employ a Lyapunov function of the form

Having

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

The application of Threshold Accepting Algorithms yields

$$P = \begin{bmatrix} 1.408 & -0.2556 \\ -0.2556 & 1.1871 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} -0.2240 & 0.7618 \\ 0.7618 & -5.0692 \end{bmatrix}$$

Therefore, $V(x)$ and $\dot{V}(x)$ are, respectively, of the form:

$$V(x) = 1.408x_1^2 - 0.5112x_1x_2 + 1.1871x_2^2$$

and

$$\dot{V}(x) = -0.2240x_1^2 + 1.5236x_1x_2 - 5.0692x_2^2$$

Their surface plots are illustrated in Figure 2 (a) and (b), respectively.

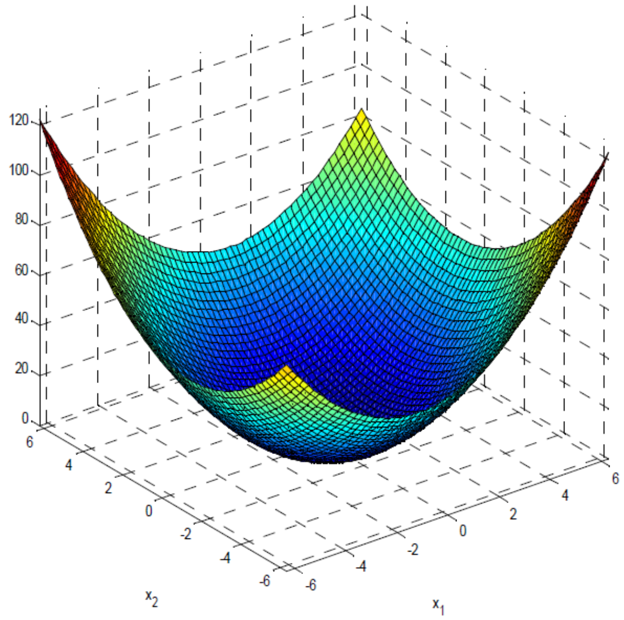


Figure 2 (a) Lyapunov Function $V(x)$ of example 1

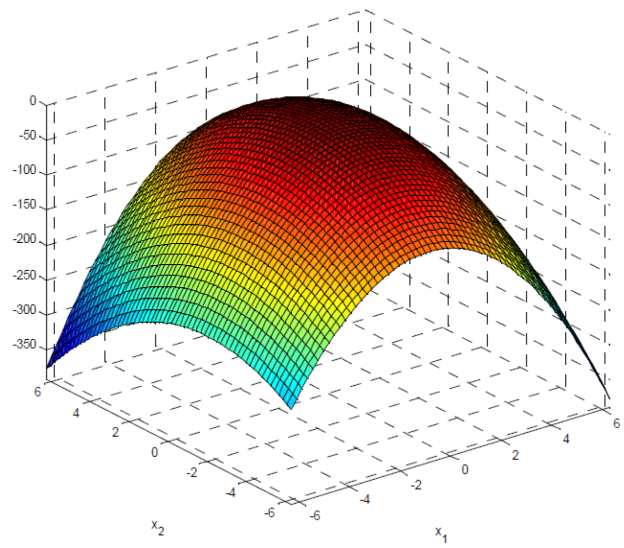


Figure 2 (b). The time derivative of the Lyapunov Function $\dot{V}(x)$ of example 1

In order to validate the above expression we have conducted a numerical simulation study. Figure 2 (a) shows clearly that $V(x) > 0$ and it is obvious in Figure 2 (b) that $\dot{V}(x) < 0$. This concludes that $V(x)$ is a valid Lyapunov Function and the asymptotic stability at the origin can be concluded.

In order to find the shape of the Region of Attraction we apply the LMI optimization. Since the degree $\delta_L$ of $\dot{V}(x)$ is 4 , we can select $\delta_s = 1$ which implies that $m = 2$ .Vectors $x^{\{\delta_V\}} = x^{\{\delta_s\}}$ and $x^{\{m\}}$ are selected as:

$$x^{\{\delta_V\}} = x^{\{\delta_s\}} = (x_1, x_2)^T$$

and

$$x^{\{\delta_m\}} = \left(x_1, x_2, x_1^2, x_1x_2, x_2^2\right)^T$$

which implies that:

$$D_f(\alpha, p_{1,2,3}) = \begin{bmatrix} -4p_1 & -3p_2 & 0 & \alpha_1 & \alpha_2 \\ -3p_2 & -2p_3 & -(p_1 - p_2) - \alpha_1 & -(p_2 - p_3) - \alpha_2 & 0 \\ 0 & -(p_1 + p_2) - \alpha_1 & 0 & 0 & \alpha_3 \\ \alpha_1 & -(p_1 + p_2) - \alpha_2 & 0 & -2\alpha_3 & 0 \\ \alpha_2 & 0 & \alpha_3 & 0 & 0 \end{bmatrix}$$

with

$$S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

$$W(S, p_{1,2,3}) = \begin{bmatrix} s_1 & s_2 & 0 & 0 & 0 \\ s_2 & s_3 & 0 & 0 & 0 \\ 0 & 0 & \mu p_1 s_1 & \mu(p_1s_2 + p_2s_1) & 0 \\ 0 & 0 & \mu(p_1s_2 + p_2s_1) & \mu(p_1s_3 + 4p_2s_2 + p_3s_1) & \mu(p_2s_3 + p_3s_2) \\ 0 & 0 & 0 & \mu(p_2s_3 + p_3s_2) & \mu p_3 s_3 \end{bmatrix}$$

and

$$W_2(S, p_{1,2,3}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_1 s_1 & (p_1s_2 + p_2s_1) & 0 \\ 0 & 0 & (p_1s_2 + p_2s_1) & (p_1s_3 + 4p_2s_2 + p_3s_1) & (p_2s_3 + p_3s_2) \\ 0 & 0 & 0 & (p_2s_3 + p_3s_2) & p_3 s_3 \end{bmatrix}$$

In this step, we define the shape of the RA with the Polynomial Lyapunov Function in $x$ we perform this step of the algorithm, by considering the best estimation of parameters $p_1, p_2, p_3$ .

The result of this step is

$$\dot{V}(x) = 1.4080\mathrm{x}_1^2 - 0.5112\mathrm{x}_1\mathrm{x}_2 + 1.1871\mathrm{x}_2^2 = 6.4628$$

and is represented in figure 3.

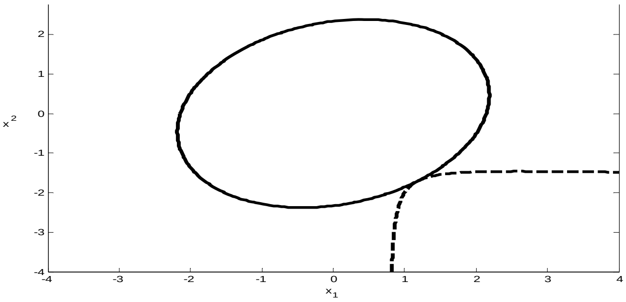


Figure 3. The Solid line indicates the boundary of the RA V(x) = 6.4628.

The dashed line indicates the constraint $\dot{V}(x) = 0$ .

*B. Example2*

Consider the Van der pol system

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 + \left(x_1^2 - 1\right)x_2$$

According to the algorithm, proposed in paragraph IV, we employ a Lyapunov Function of the form

$$V(x) = p_1x_1^2 + 2p_2x_1x_2 + p_3x_2^2$$

Having

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

The application of Threshold Accepting Algorithms return

$$P = \begin{bmatrix} 4.6 & -0.901 \\ -0.901 & 3.926 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} -1.802 & 0.227 \\ 0.227 & -6.05 \end{bmatrix}$$

Therefore, $V(x)$ and $\dot{V}(x)$ are of the form:

$$V(x) = 4.6\mathrm{x}_1^2 - 1.8020\mathrm{x}_1\mathrm{x}_2 + 3.926\mathrm{x}_2^2$$

and

$$\dot{V}(x) = -1.802\mathrm{x}_1^2 + 0.4540\mathrm{x}_1\mathrm{x}_2 - 6.05\mathrm{x}_2^2$$

whose surface plots are illustrate in Figure 4 (a) and (b), respectively.

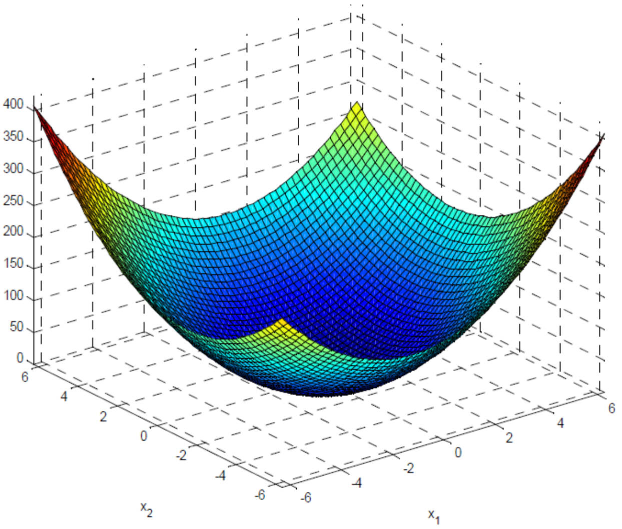


Figure 4 (a). Lyapunov Function $V(x)$ of example 2

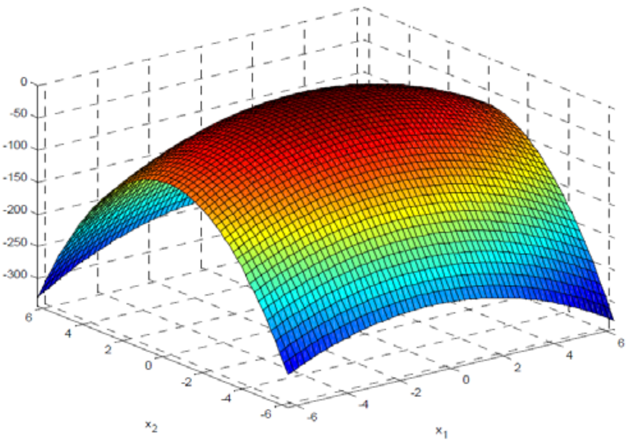


Figure 4 (b). The time derivative of Lyapunov Function $\dot{V}(x)$ of example 2

In order to validate the above expression, a numerical simulation study has been conducted. Figure 4 (a) shows clearly that $V(x)>0$ and it is obvious in Figure 4 (b) that $\dot{V}(x)<0$. This concludes that $V(x)$ is a valid Lyapunov Function and the asymptotic stability at the origin can be concluded.

In order to find the shape of the Region of Attraction we apply the LMI optimization. Since the degree $\delta_L$ of $\dot{V}(x)$ is 4 , we can select $\delta_s=1$ which implies that $m=2$ .Vectors $x^{\{\delta_V\}}=x^{\{\delta_s\}}$ and $x^{\{m\}}$ are selected such that

$$x^{\{\delta_V\}}=x^{\{\delta_s\}}=\left(x_1,x_2\right)^T$$

and

$$x^{\{\delta_m\}}=\left(x_1,x_2,x_1^2,x_1x_2,x_2^2\right)^T$$

which implies that:

$$D_f\left(\alpha,p_{1,2,3}\right)=\begin{bmatrix} 2p_2 & -\left(p_1+p_2-p_3\right) & 0 & \alpha_1 & \alpha_2 \\ -\left(p_1+p_2-p_3\right) & -2\left(p_2+p_3\right) & -\alpha_1 & -\alpha_2 & 0 \\ 0 & -\alpha_1 & 0 & p_2 & \alpha_3 \\ \alpha_1 & -\alpha_2 & p_2 & -2\left(\alpha_3-p_3\right) & 0 \\ \alpha_2 & 0 & \alpha_3 & 0 & 0 \end{bmatrix}$$

with

$$S=\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$$

$$W\left(S,p_{1,2,3}\right)=\begin{bmatrix} s_1 & s_2 & 0 & 0 & 0 \\ s_2 & s_3 & 0 & 0 & 0 \\ 0 & 0 & \mu p_1s_1 & \mu\left(p_1s_2+p_2s_1\right) & 0 \\ 0 & 0 & \mu\left(p_1s_2+p_2s_1\right) & \mu\left(p_1s_3+4p_2s_2+p_3s_1\right) & \mu\left(p_2s_3+p_3s_2\right) \\ 0 & 0 & 0 & \mu\left(p_2s_3+p_3s_2\right) & \mu p_3s_3 \end{bmatrix}$$

and

$$W_2\left(S,p_{1,2,3}\right)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_1s_1 & \left(p_1s_2+p_2s_1\right) & 0 \\ 0 & 0 & \left(p_1s_2+p_2s_1\right) & \left(p_1s_3+4p_2s_2+p_3s_1\right) & \left(p_2s_3+p_3s_2\right) \\ 0 & 0 & 0 & \left(p_2s_3+p_3s_2\right) & p_3s_3 \end{bmatrix}$$

In this step, we define the shape of the DA with the Polynomial Lyapunov Function in $x$. We perform this step of the algorithm, by considering the best estimation of the parameters $p_1,p_2,p_3$.

The result of this step is

$$V(x)=4.6x_1^2-1.8020x_1x_2+3.926x_2^2=7.7096$$

and is represented in figure 5.

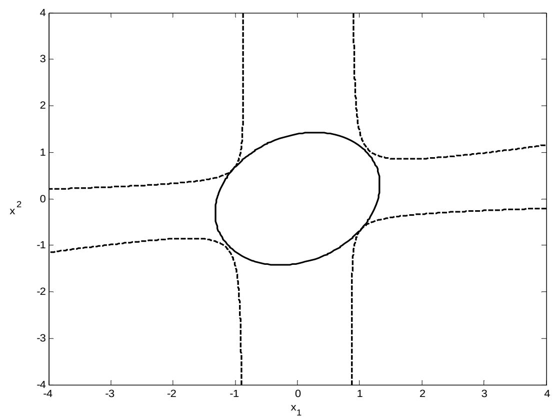


Figure 5. The Solid line indicates the boundary of the RA $V(x)=7.7096$ .

The dashed line indicates the constraint $\dot{V}(x)=0$ .

## VI. CONCLUSION

The problem of determining a Lyapunov Function candidate is investigated. We are particularly interested in autonomous nonlinear systems which represent a larger class of physical nonlinear dynamic systems. An LMI-based optimization strategy is utilized in order to compute the Region of Attraction (RA). The main contribution consists in determining an explicit RA by using a parameterized Lyapunov Function. The parameters are computed by combining Threshold Accepting Algorithms (TAA) and LMI. Two Examples have illustrated the efficiency of the established results.

## REFERENCES

[1] A. M. Lyapunov, "General Problem of Stability of Motion." Markov: *Math. Soc. 1892. Published in Collected Papers, Moscow – Leningrad, Academy of Science, USSR*, pp. 5 – 263, 1956.

[2] A. Papachristodoulou, S. Prajna, "On the Construction of Lyapunov Functions using the Sum of Squares Decomposition." *Proc. Of IEEE CDC*, 2002.

[3] A. Papachristodoulou, S. Prajna, "The Construction of Lyapunov Functions using the Sum of Squares Decomposition." *In Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 3482–3487, 2002.

[4] A. A. Martynyuk, V. I. Slynko, "Solution of the Problem of Constructing Lyapunov Matrix Functions for a class of Large Scale Systems." *Nonlinear Dynamics and Systems Theory,* Vol. 1, No. 2, pp. 193 – 203, 2001.

[5] A. Vanelli and M. Vidyasagar, "Maximal Lyapunov Function and Domain of Attraction for Autonomous Nonlinear Systems." *Automatica*, Vol.21, pp. 69-80 1985.

[6] C. Navarro Hernandez, "Linearization Methods and Control of Nonlinear Systems-Two Cases." *Monach University, Australia, ECSE Seminar on Friday 2nd* March, 2005.

[7] C. Navarro Hernandez and S. P. Bank, "A generalization of Lyapunov Equation to Nonlinear Systems." *Nocols, Stuttgart, Germany,* 2004.

[8] C. Bank, "Searching for Lyapunov Functions using Genetic Programming." *Nolcos, Stuttgart, Germany*, 2004.

[9] E.J.Davison and E.M.Kurak, "A computational method for determining quadratic Lyapunov function for nonlinear Systems." *Automatica,* Vol.7, pp.627-636, 1971.

[10] F. Hamidi and H. Jerbi, "On the estimation of a maximal Lyapunovfunction and domain of attraction determination via a genetic algorithm." *The 6th International Multi- Conference on Systems, Signals and Devices. Djerba, Tunisia.* March 23-26, 2009.

[11] F. Hamidi, H. Jerbi, W. Aggoune, M. Djemai and M.N. Abdelkrim, "Enlarging region of attraction via LMI-based approach and Genetic Algorithm." *The 1st International Conference on Communication Computing and Control Applications Hammamet, Tunisia,* March 3-5, 2011.

[12] G. Chesi, A. Garulli, A Tesi and A. Vicino, "Homogeneous Polynomial Forms for Robustness Analysis of Uncertain Systems." *Springer Verlag,* 2009.

[13] G. Chesi, "Estimating the domain of attraction via union of continuous families of Lyapunov estimates." Systems and Control Letters, Vol. 56, pp. 326-333, 2007.

[14] G. Chesi, "Computing output feedback controllers to enlarge the domain of attraction in polynomial systems." IEEE Transaction on Automatic Control, Vol. 49, pp. 1846-1850, 2004.

[15] G. Chesi, A. Garulli, A Tesi and A. Vicino, "Characterizing the solution set of polynomial systems in terms of homogeneous forms: an LMI approach." *International Journal of Nonlinear and Robust Control,* Vol.13, no. 13, pp. 1239-1257, 2003.

[16] G. Chesi, A. Garulli, A Tesi and A. Vicino, "Solving quadratic distance problems: an LMI approach." *IEEE Transaction on Automatic Control,* Vol. 13, pp. 200-212, 2003.

[17] G. Chesi, A. Garulli, A Tesi and A. Vicino, "Homogeneous Lyapunov functions for systems with structured uncertainties." *Automatica,* Vol.39, no. 6, pp. 1027-1035, 2003.

[18] G. Duech and T. Scheuer, "Threshold accepting: A general purpose optimization algorithm appearing superior to simulated annealing," *Journal of computational Physics,* 90, pp. 161-175, 1990.

[19] H.Khalil, "Nonlinear Systems," *Prentice Hall, Third Edition*, 2002.

[20] H.Khalil, "Nonlinear Systems." *McMillan Publishing Company, New York,* 1992.

[21] J. La Salle, S. Lefschetz, "Stability by Lyapunov's Direct Method with Applications." *Unpublished. New York, Academic Press,* 1961.

[22] K. Hedrih, "Nonlinear Dynamics and Alexander Mikhailovich Lyapunov." *Scientific Technical Review; Mechanics, Automatic Control and Robotics,* Vol. 6, No. 1, pp. 211-218, 2007.

[23] R. Genesio, M. Tartaglia, and A. Vicino, "On the estimation of asymptotic stability regions: state of the art and new proposals." *IEEE Transaction on Automatic Control,* Vol. AC 30, no. 8, pp. 747-755, 1985.

[24] V. Lakshmikantham, V. M. Matrosov, S. Sivasundaram, "Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems." *Netherlands,* 1991.

[25] Y. Fujisaki, and R. Sakuwa, "Estimation of asymptotic stability regions via homogeneous polynomial Lyapunov functions." *International Journal Control,* Vol. 79, no. 6, pp. 617-623, 2006.