Simulation Models and Algorithms based on Stochastic Jump Processes with Time Substitution

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***Abstract* – In this paper, we propose new simulation models and algorithms designed for construction Levy-type jump stochastic processes. Our aim is to present new simulation models and methods based on time substitution, which are faster and less complex relative to known. A time substitution (means the change of stochastic process modelling time), named also "subordination" allows to simplify Levy type process simulation and get more effective calculation structure. The subordination is considered in three ways: the determined substitution of time; the determined substitution of time for compound Poisson process and stochastic substitution of time. The algorithms of simulation of Levy process in cases of Levy measure is finite and Levy measure is infinite are developed.**

Keywords – Jump Stochastic Process, Levy Stochastic Process, Stochastic Process Simulation, Time Substitution in Stochastic Process Simulation

I. INTRODUCTION

The work has the following structure: the second section describes the basic mathematical facts which can have practical use for construction of models of various natural phenomena, such as diffusion of streams in porous media and plasma, laser cooling, molecular collisions, long-term climate changes, movement of molecules in rarefied gas, interference in communication channels, models of teletraffic, fluctuations of profitability of financial instruments, solutions of applied problems arising in financial mathematics, etc. Only models with stationary and independent increments are considered.

From all the variety of literature on stochastic processes [1-8] the authors, on the one hand, have selected the facts which from their point of view are the most important for construction of adequate mathematical models of solvable problems. On the other hand, the authors restricted themselves, so that mathematical models should be embodied into computing algorithms, and then in the software. Therefore, the models considered in the work are a compromise between their complexity and realizability. In this connection Levy processes are of particular interest because they are generated by one-dimensional distributions (univariate distributions) of a special kind. Depending on the kind of one-dimensional distribution all the variety of Levy processes is formed and they are a convenient modeler from the point of view of computing, in comparison with the processes of nonlinear diffusion.

There exist two alternative methods for solving the problems connected with the calculation of functionals on trajectories of stochastic process. The first method is the Monte-Carlo method, and the second is the numerical solution of the integro-differential equations in partial derivatives.

For the realization of Monte-Carlo method the crucial problem is the working out of effective numerical procedures of generation of the trajectories of stochastic processes or the time series. The fullest review of existing methods of generation for Levy processes is presented in work [9].

The main objective attained in the third section of the work is the working out of time series generators allowing the calculation of various functionals from time series using Monte-Carlo method. The basic difficulty of analytical character when constructing a Levy process generator is that for the majority of Levy processes the law of distribution of increments is not known in an explicit form. In this connection the authors consider to be justified the splitting of the problem of generation of trajectories of Levy process into some subtasks: 1) simulation of compound Poisson process, 2) approximation of Levy process by compound Poisson process.

The trajectories of compound Poisson process are piecewise constant with final number of jumps in the limited interval; therefore these trajectories can be modeled rather exactly. We offer algorithms that use the fact that there can be one jump at most in a sufficiently small time interval. Besides, we consider the algorithm which takes into account the possibility of the determined replacement (change) of time. The elementary approximation is that the modulo jumps not exceeding ε are removed. Such approximation converges slowly in case of high concentration of jumps in the neighborhood of zero. For the description of modulo jumps less than ε it is possible to use appropriately normalized Brownian motion.

II. BASIC SYMBOLS, DEFINITIONS AND CLASSIFICATION OF PROCESSES

Let's consider (Ω, F, P) to be probabilistic space and T - time scale. The time scale is either a set of integer nonnegative numbers N or $T = \{0,1,...,T\}$; $\boldsymbol{T} = R^{+}$, or $\boldsymbol{T} = [0,T]$.

Stochastic function or process [9] is the name for representation of $X : T \times \Omega \to R$, which characteristic is that for all t $X(t,\omega)$ it is a random variable and for all ω $X(t,\omega)$ it is a Borelean function.

Filtration is the name of sequence $(F_t)_{t \in T}$ σ that is the subalgebra of the algebra F , having the properties:

а) $\tau < t \Rightarrow F_\tau \subseteq F_t$;

б) $\lim_{\tau \downarrow t} F_\tau = F_t$;

в) $F = \bigcup_{t=0}^{\infty} F_t$, if time scale is infinite (1)

$F = F_T$, if time scale is finite.

Filtration is the reflection of the dynamics of the investigated phenomenon modeled by means of the process. In this connection it is possible to consider F_t as the information accessible to "observer" up to the time moment t inclusive.

Process X is called consistent with a filtration, if for all t $X(t,\bullet)$ is a measurable function relative to F_t. Any process X is consistent relative to natural filtration $F_t = \sigma\{X_\tau, 0 \le \tau \le t\}$.

Let's consider the classification of processes on the basis of their special properties.

Stationary in narrow sense. Property of stationary in the narrow sense consists in the following: for all n, h and for all $t_1 < t_2 < .... < t_n$

$$
\begin{aligned}
&Law\left(X(t_1,\omega), X(t_2,\omega),...,X(t_n,\omega)\right) = \\
&= Law\left(X(t_1+h,\omega), X(t_2+h,\omega),...,X(t_n+h,\omega)\right).
\end{aligned} \qquad (2)
$$

That is, invariance relatively to shift in time takes place. In particular it means that the probabilistic average and dispersion X_t are the constants which are not dependent on time.

Stationarity in a broad sense means that covariation $Cov(X_t, X_\tau) = R(|t-\tau|)$. The notation $X_t = X(t,\omega)$ is used here and below. In particular, $DX_t = Cov(X_t, X_t) = R(0)$. It is natural that stationarity in a broad sense follows from stationarity in a narrow sense.

The following important quality of stochastic processes is its Markovian property which is reflected through conditional laws of distribution as follows: $Law\left(X_t / (X)_{s \le \tau}\right) = Law\left(X_t / X_\tau\right)$ for all $\tau < t$.

III. THE BASE OF LEVY PROCESSES

The processes having these useful qualities are processes with independent and stationary increments, that is, they are Levy processes.

Processes called Levy processes [1,2,9] have the following properties:

1) $X_0 = 0$;

2) $\forall n$ and $\forall t_1 < t_2 < ... < t_n$ random variables $X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent;

3) $\forall s < t$ and $\forall h$ $Law(X_t - X_s) = Law(X_{t+h} - X_{s+h})$;

4) $\forall \varepsilon > 0$ $\lim_{t \to s} P(|X_t - X_s| > \varepsilon) = 0$;

5) the trajectories are semi-continuous at the right and have limits at the left.

From the second and the third property it follows directly that the process is completely described by the one-dimensional infinitely decomposable distribution law $Law(X_t)$, the characteristic function of which is $\varphi_t(\theta) = E\exp(i\theta X_t) = \exp(t\psi(\theta))$. The function $\psi(\theta)$ is called characteristic exponent or cumulant, and

$$
\begin{aligned}
&\psi(\theta) = i\theta m - \frac{\sigma^2}{2}\theta^2 + \\
&+ \int_{-\infty}^{\infty} \left(\exp(i\theta x) - 1 - i\theta x I_{(-1,1)}(x)\right)\nu(dx).
\end{aligned} \qquad (3)
$$

Measure $\nu(dx)$ is called Levy measure and it should comply with the following contingencies $\int_{-\infty}^{\infty} (x^2 \wedge 1)\nu(dx) < \infty$. If to strengthen the condition for Levy measure, namely to demand for

$\int_{-\infty}^{\infty}(|x|\wedge 1)\nu(dx)<\infty$, then integrals $\int_{-\infty}^{\infty}(\exp(i\theta x)-1)\nu(dx)$, $\int_{-1}^{1}x\nu(dx)$ exists, and cumulant will be :

$$\psi(\theta)=i\theta m-\frac{\sigma^2}{2}\theta^2+\int_{-\infty}^{\infty}(\exp(i\theta x)-1)\nu(dx) \qquad (4)$$

with the other value of m.

The use of Levy processes is sufficient for the majority of problems of mathematical modeling (and simulation) with cumulant (4). We will consider two cases. In the first case a Levy measure is finite – $\int_{-\infty}^{\infty}\nu(dx)<\infty$. In this the cumulant (4) can be presented in the form:

$$\psi(\theta)=i\theta m-\frac{\sigma^2}{2}\theta^2+\lambda\int_{-\infty}^{\infty}(\exp(i\theta x)-1)F(dx),$$

where $\lambda=\int_{-\infty}^{\infty}\nu(dx)$ and $F(dx)=\frac{\nu(dx)}{\lambda}$ is the probability measure.

In this case Levy process with cumulant (4) $X_t=mt+\sigma W_t+Y_t$, where W_t is the standard Wiener process, Y_t is the compound Poisson process with the intensity of jumps λ and with the distribution of the jump size F.

In the second case the Levy measure is infinite: $\int_{-\infty}^{\infty}\nu(dx)=\infty$. Suppose $\varepsilon>0$ is sufficiently small positive quantity. Then $\int_{[(-\infty,-\varepsilon)\cup(\varepsilon,+\infty)]}\nu(dx)<\infty$. Levy process with cumulant

$$\psi_\varepsilon(\theta)=i\theta m-\frac{\sigma^2}{2}\theta^2+\int_{[(-\infty,-\varepsilon)\cup(\varepsilon,+\infty)]}(\exp(i\theta x)-1)\nu(dx)$$

is presented as a sum $X_t^{(\varepsilon)}=mt+\sigma W_t+Y_t^{(\varepsilon)}$, where $Y_t^{(\varepsilon)}$ is the compound Poisson process. $\forall t$ $X_t^{(\varepsilon)}$ at least is weakly convergent to X_t. More detailed description is in [9].

To present Levy process we can use the fact that Levy measure σ-is finite, i.e, there exists partition $R=\bigcup_{i\geq 1}B_i$, with $\nu(B_i)<\infty$. Let's consider the sequence of probability measures $F_i: F_i(B)=\frac{\nu(B\cap B_i)}{\nu(B_i)}$. For each probability measure there exists the compound Poisson process $X^{(i)}$ with intensity $\lambda_i=\nu(B_i)$ and jump distribution $F_i(dx)$. Then $X=mt+\sigma W_t+\sum_{i=1}^{\infty}X^{(i)}$.

For the case when $\int_{-\infty}^{\infty}(x\wedge 1)\nu(dx)=\infty$. The integral is presented as

$$\int_{[(-\infty,-\varepsilon)\cup(\varepsilon,+\infty)]}\left(\exp(i\theta x)-1-i\theta xI_{(-1,1)}(x)\right)\nu(dx)=$$

$$=\int_{(-\infty,-1]\cup[1,+\infty)}(\exp(i\theta x)-1)\nu(dx)+$$

$$+\int_{(-1,-\varepsilon]\cup[\varepsilon,1)}(\exp(i\theta x)-1-i\theta x)\nu(dx).$$

The first summand corresponds to compound Poisson process with intensity of jumps $\lambda=\int_{(-\infty,-1]\cup[1,+\infty)}\nu(dx)$ and jumps distribution $F(dx)=I_{\{|x|\geq 1\}}(x)\frac{\nu(dx)}{\lambda}$ (process Y). The second summand corresponds to compensated compound Poisson process with intensity $\lambda=\int_{(-1,-\varepsilon]\cup[\varepsilon,1)}\nu(dx)$ and jumps distribution $F(dx)=I_{\{|x|<1\}}(x)\frac{\nu(dx)}{\lambda}$ (process $Y^{(\varepsilon)}$).

Process $X_t^{(\varepsilon)}=mt+\sigma W_t+Y_t+Y_t^{(\varepsilon)}$. It is natural that for all t $X_t^{(\varepsilon)}$ at least is weakly convergent to X_t.

The subsets of infinitely divisible laws of distribution are steady laws of distribution [10], for which the cumulant is

$$\psi(\theta)=\begin{cases} i\mu\theta-\sigma^\alpha|\theta|^\alpha\left(1-i\beta(Sgn\theta)tg\frac{\pi\alpha}{2}\right), & \text{если } \alpha\neq 1;\\ i\mu\theta-\sigma|\theta|\left(1+i\beta\frac{2}{\pi}(Sgn\theta)\ln|\theta|\right), & \text{если } \alpha=1.\end{cases} \qquad (5)$$

Parameters of equation (5) have the following meaning:

α – stability index ($\alpha\in(0,2]$);

μ – location parameter;

σ – scale parameter ($\sigma>0$);

β – parameter of skewness of relative distribution function ($\beta\in[-1,1]$).

Levy process with cumulant (5) is α-stable process, that is for all $a>0$ $Law(X_{at})=Law\left(a^{1/\alpha}X_{at}+Dt\right)$. With $D=0$ Levy process is called strictly stable. Let's compare cumulant (5) with cumulant (4), with stability index $\alpha=2$ parameter β is random, for example, is equal to zero, and Levy process in this case is the linear transformation of Brownian motion with drift ; with stability index $\alpha\neq 2$ parameter σ in equations (3), (4) is equal to zero, that means the absence of Brownian component in Levy process. Suppose $\alpha\neq 2$,and we consider Levy measure in this case. In connection with this we give the theorem

Theorem 1 [9]. Suppose that $\alpha \neq 2$, then Levy measure $\nu(dx) = \frac{C_1}{x^{1+\alpha}} I_{(0,+\infty)}(x)dx + \frac{C_2}{|x|^{1+\alpha}} I_{(-\infty,0)}(x)dx$,

where $C_1 \geq 0, C_2 \geq 0, C_1 + C_2 > 0$.

And we can easily see that with $\alpha \in (0,1]$ the formula (4) is valid, and with $\alpha \in (1,2)$ the formula (3) is valid. This Levy measure is an infinite Levy measure. Hence, for stable Levy processes the expression $X_t = mt + \sigma W_t + \sum_{i=1}^{\infty} X_t^{(i)}$ or $X_t = mt + \sigma W_t + \lim_{\varepsilon \downarrow 0} Y_t^{(\varepsilon)}$ is valid.

For strict α stability of Levy process it is necessary and enough, that with $\alpha \neq 1$ parameter $\mu = 0$, and with $\alpha = 1$ parameter $\beta = 0$.

For symmetric α -steady laws of distribution $\mu = \beta = 0$. In this case the formula for cumulant (4) takes the form:

$$\psi(\theta) = -\sigma^{\alpha} |\theta|^{\alpha}. \qquad (6)$$

Levy process with cumulant (6) is strictly α - stable.

Natural generalization of stable Levy processes are processes with Levy measure $\nu(dx) = \frac{k(x)}{|x|^{1+\alpha}} dx$, where $k(x)$ is never- increasing function on the $(0,+\infty)$ interval and never- decreasing function on the $(-\infty,0)$ interval.

For the symmetric case $C_1 = C_2$ and $k(x) = k(-x)$.

In connection with the rate of decrease of tails we will pay attention to parameter α. This parameter is responsible the for rate of tail decrease since $\lim_{x\to\infty} x^{\alpha} P(X > x) = C_{\alpha} \frac{1+\beta}{2} \sigma^{\alpha}$, where $\alpha \in (0,2)$.

From this equation it follows directly that stable distributions are the distributions with heavy tails.

Among α - stable distributions with heavy tails the explicit form of laws is known only in two cases and that complicates their use.

1. Cauchy distribution $(\alpha = 1)$ with density

$$\frac{\sigma}{\pi\left((x-\mu)^2 + \sigma^2\right)}.$$

2. Levy distribution $\left(\alpha = \frac{1}{2}\right)$ with density

$$\left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp\left(-\frac{\sigma}{2(x-\mu)}\right).$$

Let's notice that the density of stable distributions can be presented in the form of series [11].

IV. GENERALIZED HYPERBOLIC DISTRIBUTIONS

The other important subclass of infinitely divisible distributions are generalized hyperbolic distributions. This class of distributions introduced in 1977 by O. Barndorff-Nielsen [9], is intermediate between stable distributions with an index $\alpha < 2$ and Gaussian distribution from the point of view of tails decrease.

From this class of distributions the following distributions are usually singled out:

1) hyperbolic distribution with density:

$$p(x,\alpha,\beta,\delta,\mu) = C(\alpha,\beta,\delta)\exp\left(-\alpha\sqrt{\delta^2 - (x-\mu)^2} + \beta(x-\mu)\right), \qquad (7)$$

where $C(\alpha,\beta,\delta) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}$, and $K_1(x)$ is modified Bessel function of the first kind with index 1.

It is supposed, that parameters satisfy to boundary conditions $\alpha > 0,\ |\beta| < \alpha,\ \delta > 0$. The replacement of random variable X by random variable $Y = (X - a)/b$ for $a \in R$ and $b > 0$ also leads to the change of density (7) for density $p(x, b\alpha, b\beta, \delta/b, (\mu - a)/b)$. Thereby, the hyperbolic distribution is invariant relative to shift and scale. In particular, if $\mu = a$ and $b = \delta$ then the density will take the form $p(x, \delta\alpha, \delta\beta, 1, 0)$.

Let's consider the rate of decrease of tails of hyperbolic distribution. Let $x \to +\infty$,

then $\exp\left(-\alpha\sqrt{\delta^2 - (x-\mu)^2} + \beta(x-\mu)\right) \sim \exp\left(-(\alpha - \beta)x\right)$,

with $x \to -\infty$.

Hence, judging from the rate of the decrease of tails the hyperbolic distribution belongs to the class of distributions with exponentially decreasing tails. It is noticed in the work [1] that hyperbolic distribution comes out as a result of the averaging of Gaussian distribution. If the random variable X is distributed as hyperbolic distribution, then

$$LawX = E_{\sigma^2} N\left(\mu + \beta\sigma^2, \sigma^2\right), \qquad (8)$$

where averaging is taken for density:

$$p(x) = \frac{\sqrt{a/b}}{2K_1\left(\sqrt{ab}\right)} \exp\left(-\frac{1}{2}\left(ax + \frac{b}{x}\right)\right),$$

$a = \alpha^2 - \beta^2, b = \delta^2$.

Let's consider the other representative of the class of generalized hyperbolic distributions – Gaussian/inverse Gaussian (*GIG*) distribution. Distribution *GIG* can be obtained by the averaging of Gaussian distribution

$$LawX = E_{\sigma^2} N\left(\mu + \beta\sigma^2, \sigma^2\right) \qquad (9)$$

for the measure with density $p(x)=\sqrt{\frac{b}{2\pi}}e^{\sqrt{ab}}\frac{1}{x^{3/2}}\exp\left(-\frac{1}{2}\left(ax+\frac{b}{x}\right)\right)$ (Distribution with this density is called inverse Gaussian distribution). We will consider generating function *GIG* of distribution

$$Ee^{\lambda X}=\exp\left(\delta\left(\sqrt{\alpha^2+\beta^2}-\sqrt{\alpha^2-(\beta+\lambda)^2}\right)+\lambda\mu\right).$$

It is easy be see that the sum of independent *GIG*-distributed random variables with the same, but probably different μ and δ again has *GIG* distribution with the same α and β and with $\mu=\sum_{i=1}^{m}\mu_i,\delta=\sum_{i=1}^{m}\delta_i$. The similar property for hyperbolic distributions is absent.

The density of *GIG* distribution:

$$p(x,\alpha,\beta,\delta,\mu)=C(\alpha,\beta,\delta)\left[q\left(\frac{x-\mu}{\delta}\right)\right]^{-1}\cdot \cdot K_1\left(\alpha\delta q\left(\frac{x-\mu}{\delta}\right)\right)\exp(\beta(x-\mu)). \quad (10)$$

In the formula (10) $C(\alpha,\beta,\delta)=\frac{\alpha}{\pi}e^{\delta\sqrt{\alpha^2-\beta^2}}$, $q(x)=\sqrt{1+x^2}$.

V. Time substitution

In this section the processes which are obtained from the standard Levy processes by time substitution will be considered.

The determined substitution of time. Suppose that $\phi(t)$ is the increasing function $\phi(0)=0$. Let Y_t to be Levy process. We will define a new process $X_t=Y_{\varphi(t)}$. We will calculate $E\exp(i\theta X_t)=E\exp(i\theta Y_{\varphi(t)})=\exp(\varphi(t)\psi(\theta))$. From this it follows that X will be Levy process in the only case when $\varphi(t)=at$ $a>0$.

*The determined substitution of time for compound Poisson process***.** We will present the compound Poisson process as a point process. Suppose that the sequence ξ consists of independent equally distributed random variables with $P_\xi(\xi_i\in dx)=F_\xi(dx)$, and the sequence Δ consists of independent equally distributed random variables with $P_\Delta(\Delta_i\in dx)=F_\Delta(dx)$. Let us assume that $\tau_0=0$, $\tau_i=\tau_{i-1}+\Delta_i$. We will define the stochastic process:

$$X_t=\sum_{i\ge1}\xi_i I(\tau_i\le t) \quad (11)$$

The process X_t is a purely jump process with the random variable of jump and the intensity of jumps defined as F_Δ .

The following theorem is valid.

Theorem 2 [2]. The process X_t defined by formula (11), will be Levy process in the only case when F_Δ is exponential distribution.

Let's notice that in this case the process X_t is the compound Poisson process and it can be presented as:

$$X_t=\sum_{i=1}^{N_t}\xi_i , \quad (12)$$

where N_t is the Poisson process.

Suppose Y_t is the compound Poisson process.

Let's enter into consideration the new process $X_t=Y_{\phi(t)}$. Time substitution will lead to the change of the formula (11)

$$X_t=Y_{\phi(t)}=\sum_{i\ge1}\xi_i \mathrm{I}\{\tau_i\le\phi(t)\} \quad (13)$$

Let's consider the sequence Δ for process X . We designate $\overline{\Delta}_k=\varphi(\tau_{k-1}+\Delta_k)-\varphi(\tau_{k-1})$. The sequence $\overline{\Delta}$ generating the sequence for compound Poisson process is Y . We will calculate the conditional probability $P(\Delta_k\le x/z_{k-1})$ $(\tau_{k-1}=z_{k-1})$. The desired probability

$$P(\Delta_k\le x/z_{k-1})=$$
$$=P(\varphi(z_{k-1}+\Delta_k)-\varphi(z_{k-1})\le\varphi(x+z_{k-1})-\varphi(z_{k-1}))=$$
$$=P(\overline{\Delta}_k\le\varphi(x+z_{k-1})-\varphi(z_{k-1})).$$

As Y is the compound Poisson process, then $P(\Delta_k\le x/z_{k-1})=1-\exp(-\lambda(\varphi(x+z_{k-1})-\varphi(z_{k-1})))$, and the density of conditional distribution (under the condition of differentiability of function ϕ)

$$p_{\Delta_k}(x_k\mid x_1,x_2,...,x_{k-1})= \lambda\phi'(z_{k-1}+x_k)\exp(-\lambda(\phi(z_{k-1}+x_k))-\phi(z_{k-1})), \quad (14)$$

where $z_r=\sum_{i=1}^{r}x_i$. Note that Δ_k depends on $\sum_{i=1}^{k-1}\Delta_i$. Independence is possible only when $\varphi(t)=at$. In this case $p_{\Delta_k}(x_k\mid x_1,x_2,...,x_{k-1})=p_{\Delta_k}(x_k)=a\lambda\exp(-a\lambda x_k)$.

Hence, process X is the compound Poisson process. It is well coordinated with the previous statement. Other ways of generation of stochastic processes on the basis of compound Poisson process will be considered in the following section in connection with simulation modeling.

Stochastic substitution of time. The second way of obtaining a new process is the subordination of reference process.

Definition 1.

Process of Levy is called the subordinator if values of process belong to $[0,+\infty)$ and the process trajectory is almost sure increasing. It is obvious that the subordinator is the process with limited variation. Hence the cumulant of

the subordinator:
$\psi(\theta) = i\theta m + \int_0^\infty (\exp(i\theta x) - 1)\nu(dx)$, $\int_{-\infty}^0 \nu(dx) = 0$. Let's consider the Laplace conversion of subordinator:

$$E\exp(-\lambda T_t) = \exp(-t\Phi(\lambda)). \quad (15)$$

Function $\Phi(\lambda)$ is called Laplace exponent. As the function $\psi(\theta)$ can be analytically prolonged from a real line to the upper half plane, and

$$\Phi(\lambda) = \lambda m - \int_0^\infty (\exp(-\lambda x) - 1)\nu(dx) \quad (16)$$

Here are the most known subordinators [9].

The Poisson subordinator. It is easy to understand that Poisson process and compound Poisson process with positive jumps are subordinators.

Stable process. Stable Levy process with cumulant $\psi_T(u) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (\exp(iux) - 1)\frac{dx}{x^{1+\alpha}}$, where $0 < \alpha < 1$, Γ - Gamma function is the subordinator. Parameters μ, β, σ in the formula (5) are of the form: $\mu = 0, \beta = 1, \sigma^\alpha = \cos\frac{\alpha\pi}{2}$, with $\alpha = \frac{1}{2}$ density of distribution $p_{T_t}(x) = \frac{t}{2\sqrt{\pi}} x^{-3/2} \exp(-t^2/4x)$. This subordinator is called Levy subordinator and has the following interpretation: $T_t = \inf\left\{s : W_s = \frac{t}{\sqrt{2}}\right\}$. Laplace exponent of stable subordinator: $\Phi(\lambda) = \lambda^\alpha$.

We can call stochastic process $T_t = \inf\{s : X_s = t\}$ a generalized conversion process X with continuous trajectories. Let's consider stochastic process $C_t = \gamma t + \sigma W_t$ with $\gamma > 0$. The inverse Gaussian subordinator is defined as the generalized conversion of this process: $T_t = \inf\{s : C_s = \delta t\}$. The density is known

$$p_{T_t}(x) = \frac{\delta t}{\sqrt{2\pi}} e^{\gamma\delta t} \frac{1}{x^{3/2}} \exp\left(-\frac{1}{2}\left(\gamma^2 x + \frac{t^2\delta^2}{x}\right)\right).$$

Gamma subordinator. Process T_t is called Gamma subordinator, for which the density of distribution has the form: $p_{T_t}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} \exp(-bx)$.

Let's consider $X_t = Y_{T_t}$, where T_t is subordinator, Y_t is Levy process. The following theorem is valid.

Theorem 3 [9]. With independence of processes Y and T process X is Levy process. And process cumulant X - $\psi_X = \psi_T \circ (-i\psi_Y)$. And process cumulant X:

$$\psi_X(u) = b\psi_Y(u) + \int_0^\infty (\exp(iuy) - 1) m_{X,T}(dy);$$

where $m_{X,T}(dy) = \int_0^\infty F_{X_t}(dy)\lambda_T(dt)$; a $F_{X_t}(dy) = P(X_t \in dy)$, $\lambda_T(dt)$ is Levy measure of subordinator.

Subordination of compound Poisson process

Next theorem is valid.

Theorem 4. If Y is the compound Poisson process, then X is the compound Poisson process too.

Indeed, suppose the reference process is the compound Poisson process. Suppose $\overline{\Delta}$ is the generating sequence for process Y. We calculate the probability

$$F_{\Delta_i}(x) = P(\Delta_i \le x) = EP\left(\Delta_i \le x / T_{x+\tau_{i-1}} - T_{\tau_{i-1}}\right).$$

Conditional probability

$$P\left(\Delta_i \le x / T_{x+\tau_{i-1}} - T_{\tau_{i-1}}\right) =$$
$$= P\left(T_{\tau_{i-1}+\Delta_i} - T_{\tau_{i-1}} \le T_{\tau_{i-1}+x} - T_{\tau_{i-1}} / T_{x+\tau_{i-1}} - T_{\tau_{i-1}}\right) =$$
$$= P\left(\overline{\Delta}_i \le T_{\tau_{i-1}+x} - T_{\tau_{i-1}} / T_{x+\tau_{i-1}} - T_{\tau_{i-1}}\right).$$

Since Y is the compound Poisson process, then

$$P\left(\overline{\Delta}_i \le x / T_{x+\tau_{i-1}} - T_{\tau_{i-1}}\right) = 1 - \exp\left(-\lambda\left(T_{x+\tau_{i-1}} - T_{\tau_{i-1}}\right)\right).$$

From here the desired probability

$$F_{\Delta_i}(x) = P(\Delta_i \le x) = 1 - E\exp\left(-\lambda\left(T_{x+\tau_{i-1}} - T_{\tau_{i-1}}\right)\right) = 1 - E\exp(-\lambda T_x). \quad (17)$$

From (15) and (17) it follows that the desired probability

$$F_{\Delta_i}(x) = P(\Delta_i \le x) = 1 - \exp(-x\Phi(\lambda)) \quad (18)$$

From (18) it follows that the series Δ consists of independent random variables equally distributed by the demonstrative law of distribution. Hence, the subordination of compound Poisson process leads to compound Poisson process with replacement of λ for $\Phi(\lambda)$.

As the subordination of compound Poisson process does not exceed the boundaries the compound Poisson process, we refer to Wiener component of Levy process.

Subordinations of Wiener process

Let's consider the family of Levy processes which are defined by the subordination of Wiener process.

1. *The subordination α is a steady subordinator*. Let $Y_t = \sqrt{2}W_t$. The cumulant is $\psi_Y(\theta) = \theta^2$. From this and from $\Phi_T(\lambda) = \lambda^\alpha$ it follows that process cumulant $X_t = Y_{T_t}$ has the form: $\psi_X(\theta) = -|\theta|^{2\alpha}$. That is X_t process is symmetric 2α - stable process. On the basis of process X_t it is possible to obtain process $C_t = at + bX_t$ with cumulant $\psi_C(\theta) = ia - |b\theta|^{2\alpha}$. As the result, the family of

stable Levy processes with skewness parameter $\beta = 0$ can be obtained.

2. *Subordination by Gamma subordinato*r. Suppose T_t is the gamma subordinator with parameters a and b Gamma distribution, then $X_t = W_{T_t}$ with Laplace exponent $\Phi_X(\lambda) = a\ln\left(1+\frac{\lambda^2}{2b}\right)$. In the literature [9] the process has the name *variance-gamma* and can be presented in the form of a difference of two independent gamma subordinators with parameters a and $\sqrt{2b}$: $X = L - N$ that allows to use it effectively both in economic applications, and in modeling of information systems. The Levy measure of the process has the form:

$$\nu(dx) = \frac{a}{|x|}\left(\exp\left(\sqrt{b}xI_{\{(-\infty,0)\}}(x)\right) + \exp\left(-\sqrt{b}xI_{\{(-\infty,0)\}}(x)\right)\right)dx .$$

Simple generalization leads to Levy measure

$$\nu(dx) = \frac{a}{|x|^{1+\alpha}}\left(\exp\left(b_1xI_{\{(-\infty,0)\}}(x)\right) + \exp\left(-b_2xI_{\{(-\infty,0)\}}(x)\right)\right)dx ,$$

where $b_1 \ge 0, b_2 \ge 0$. With $b_1 = b_2 = 0$ the process becomes stable.

3. Gaussian/inverse Gaussian process. Gaussian/inverse Gaussian process is obtained as a result of the subordination of process $Y_t = \sigma W_t + \beta t$ and has the form: $X_t = Y_{T_t} + \mu t$. The subordinator is the inverse Gaussian subordinator. The distribution density is given in formula 10.

VI. SIMULATION OF LEVY PROCESSES

A. *Levy measure is finite*

As it has been shown in the previous section, the time series for Levy process is represented in the form of the sum of a linear trend with the sum of independent and identically distributed random variables ξ_i with characteristic function $\exp(h\psi(\theta))$ where the cumulant $\psi(\theta)$ has the form (3). The random variable ξ_i can be presented in the form of the sum of a normal random variable and a jump random variable. Generators of a normal random variable are widely presented in the literature. Therefore we will consider a jump random variable.

Generators of the jump component of Levy process are considered here and below.

For the final Levy measure ε_i are distributed according to the compound Poisson law. Also we suppose that Levy measure is known.

Algorithm 1 of ε generation.

Procedure $CP(\lambda, \zeta, F, \varepsilon)$.

1. $n := \left]\frac{\lambda^2}{\zeta}\right[$, where $]x[$ - minimal integer, that meets inequality: $]x[\ge x$.
2. $\varepsilon := 0$, $i := 1$
3. We generate a random number ξ - evenly distributed on the interval $[0,1]$
4. We generate a random number η with distribution law F
5. $\varepsilon := \varepsilon + I_{\{\xi \le \lambda/n\}}\eta$
6. $i := i+1$
7. If $i \le n$, then 3
8. End of procedure.

Comment. Parameter $\lambda = \int_{-\infty}^{\infty}\nu(dx)$. $F(dx) = \frac{\nu(dx)}{\lambda}$ ζ is the parameter setting the accuracy of calculations, and the accuracy of calculations is defined by the residual dispersion at the approximation of Poisson law by the binomial law of distribution: $D_{ocm} = \frac{\lambda^2}{n}$.

Algorithm 2 of generation of compound Poisson process.

Procedure $Tr(T, N, \lambda, \zeta, F, X)$

1. $i := 1,\ h := T/N,\ X_0 := 0,\ \mu := h\lambda$
2. $CP(\mu, \zeta, F, \varepsilon)$
3. $X_i := X_{i-1} + \varepsilon$
4. $i := i+1$
5. If $i \le N$, then 2
6. End of Procedure.

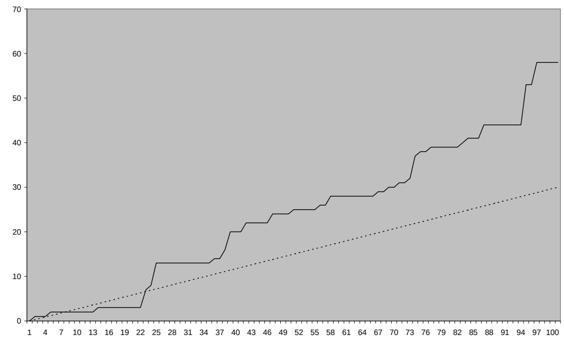

Figure 1. Generation of 100 values of compound Poisson process on simulation interval [0, 1], F - geometrical distribution with parameter $q = 0.6$; intensity $\lambda = 20$

Figure 1 presents the trajectory of compound Poisson process, a solid line, and the trend, a dotted line.

Algorithm 3 of simulating compound Poisson process with time substitution.

Suppose $X_t = Y_{\varphi(t)}$, Y is compound Poisson process. We designate through $\varepsilon_i = Y_{\varphi(t_i)} - Y_{\varphi(t_{i-1})}$. As Y is Levy process, random variables ε are independent.

Let's present the function of time substitution in the form of positive function integral: then the necessary increment for calculation of jumps rate is

$$\varphi(t+h)-\varphi(t)\approx p(t)h. \quad (19)$$

$ChTr(T,N,\lambda,\zeta,F,p,X)$

1. $i:=1,\ h:=T/N,\ X_0:=0,\ t:=0$
2. $\mu:=\lambda p(t)h$
3. $CP(\mu,\zeta,F,\varepsilon)$
4. $X_i:=X_{i-1}+\varepsilon$
5. $i:=i+1$
6. $t:=t+h$
7. if $i\le N$, then 2
8. End of procedure.

Consider the example where

$$p(x)=\begin{cases}4x,\ 0\le x\le 0.5\\ -4x+4,\ 0.5<x\le 1\\ 0,\ 1<x\end{cases}.$$

Simulation results are in Figure 2.

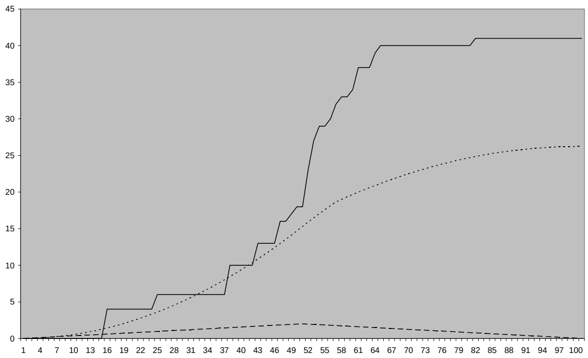

Figure 2. Generation of 100 values of compound Poisson process with time substitution on simulation interval [0, 1], F- geometrical distribution with parameter $q=0.6$; intensity $\lambda=20$

Fig. 2 presents the process trajectory as a solid line, and the trend as a dotted line. The basic difference from fig. 1 is that in the middle of the simulation interval a sharp jump of values is observed that quite corresponds to the planned effect of change of time by means of function $p(x)$ - a dotted trajectory on the graph.

B. Levy measure is infinite

General case

Let's use the representation of Levy process.

The third summand corresponds to Poisson compound process with intensity of jumps $\lambda=\int\limits_{(-\infty,-1]\cup[1,+\infty)}\nu(dx)$ and distribution of jumps $F(dx)=I_{\{|x|\ge 1\}}(x)\frac{\nu(dx)}{\lambda}$ (process Y). The fourth summand corresponds to the compensated compound Poisson process with intensity $\lambda=\int\limits_{(-1,-\varepsilon]\cup[\varepsilon,1)}\nu(dx)$ and distribution of jumps $F(dx)=I_{\{\varepsilon<|x|<1\}}(x)\frac{\nu(dx)}{\lambda}$ (process $Y^{(\varepsilon)}$). We designate Levy process as $Z_t^{(\varepsilon)}=X_t-X_t^{(\varepsilon)}$.

The cumulant of this process: $\int\limits_{-\varepsilon}^{\varepsilon}(\exp(i\theta x)-1-i\theta x)\nu(dx)$. We present the integrand in the form of Taylor series and as a result we obtain the approximate equality $\int\limits_{-\varepsilon}^{\varepsilon}(\exp(i\theta x)-1-i\theta x)\nu(dx)\approx-\frac{\theta^2}{2}\int\limits_{-\varepsilon}^{\varepsilon}x^2\nu(dx)$ that corresponds to Wiener process, therefore

$$Z_t^{(\varepsilon)}\approx\sigma_\varepsilon W_t, \quad (20)$$

where $\sigma_\varepsilon^2=\int\limits_{-\varepsilon}^{\varepsilon}x^2\nu(dx)$. The process $Y_t+Y_t^\varepsilon$ has the cumulant

$$\int\limits_{(-\infty,-\varepsilon]\cup[\varepsilon,+\infty)}(\exp(i\theta x)-1)\nu(dx)-i\theta\int\limits_{(-1,-\varepsilon]\cup[\varepsilon,1)}x\nu(dx),$$

because $\int\limits_{(-\infty,-\varepsilon]\cup[\varepsilon,+\infty)}\nu(dx)<\infty,\ \int\limits_{(-1,-\varepsilon]\cup[\varepsilon,1)}\nu(dx)<\infty$.

Consequently, we obtain the approximate equality

$$X_t\approx\overline{m}t+\overline{\sigma}W_t+U_t^{(\varepsilon)}, \quad (21)$$

where $\overline{m}=m-\int\limits_{(-1,-\varepsilon]\cup[\varepsilon,1)}x\nu(dx)$, $\overline{\sigma}=\sqrt{\sigma^2+\sigma_\varepsilon^2}$, U_t^ε Poisson compound process with (rate) intensity $\lambda=\int\limits_{(-\infty,-\varepsilon]\cup[\varepsilon,+\infty)}\nu(dx)$ and distribution of jumps $F(dx)=I_{\{|x|\ge\varepsilon\}}(x)\frac{\nu(dx)}{\lambda}$. The problem of generation of jump component reduces to the problem discussed in previous section.

C. Representation of a jump component in the form of the sum of Poisson random variables

The algorithm considered in previous section depends on the distribution of the size of jump $F(dx)$ and this is its disadvantage, because the presence of the generator for $F(dx)$ is supposed. In this section we will try to overcome this disadvantage. We will present integral $\int\limits_{(-\infty,-\varepsilon]\cup[\varepsilon,+\infty)}(\exp(i\theta x)-1)\nu(dx)$ as follows

$$\psi_\varepsilon(\theta) = \int_{(-\infty,-\varepsilon]\cup[\varepsilon,+\infty)} \left(\exp(i\theta x)-1\right)\nu(dx) =$$

$$= \int_{(-\infty,-\varepsilon]} \left(\exp(i\theta x)-1\right)\nu(dx) + \int_{[\varepsilon,+\infty)} \left(\exp(i\theta x)-1\right)\nu(dx).$$ Let's choose two constants A and B to approximate the cumulant by integrals with the desired degree of accuracy

$$\psi_\varepsilon(\theta) \approx \int_A^{-\varepsilon} \left(\exp(i\theta x)-1\right)\nu(dx) + \int_\varepsilon^B \left(\exp(i\theta x)-1\right)\nu(dx) . \quad (22)$$

Let's replace any one of integrals in (22) by final sums

$$\psi_\varepsilon(\theta) \approx \sum_{k=1}^{M} \left(\exp(i\theta a_k)-1\right)\mu_k + \sum_{k=1}^{N} \left(\exp(i\theta b_k)-1\right)\nu_k , \quad (23)$$

where $\{a_k\}$ is decomposition of interval $[A,-\varepsilon]$, $\{b_k\}$ is decomposition of interval $[\varepsilon,B]$, $\mu_k = \int_{a_{k-1}}^{a_k} \nu(dx)$, $\nu_k = \int_{b_{k-1}}^{b_k} \nu(dx)$. As a result we obtain

$$U_t^{(\varepsilon)} = \sum_{k=1}^{M} a_k Y_t^{(k)} + \sum_{k=1}^{N} b_k Z_t^{(k)} , \quad (24)$$

where $Y_t^{(k)}$ is Poisson process with intensity μ_k, $Z_t^{(k)}$ is Poisson process with intensity ν_k.

Thus, generator ε has the following form:

Algorithm 4 of generation ε.

Procedure $SP(\mu,\nu,a,b,M,N,\zeta,\varepsilon)$.

1. $k:=1, \varepsilon 1:=0$
2. $n := \left]\frac{\mu(k)^2}{\zeta}\right[$, where $]x[$ – minimal integer that meets inequality: $]x[\, \geq x$.
3. $i:=1, \varepsilon=0$
4. We generate a random number ξ evenly distributed on $[0,1]$ interval
5. $\varepsilon := \varepsilon + I_{\{\xi \leq \mu[k]/n\}}$
6. $i:=i+1$
7. If $i \leq n$, then 4
8. $\varepsilon 1 := \varepsilon 1 + a(k)\varepsilon$
9. . $k:=k+1$.
10. If $k \leq M$, then 2
11. $k:=1, \varepsilon 2:=0$
12. $n := \left]\frac{\nu(k)^2}{\zeta}\right[$, where $]x[$ is minimal integer, that meets inequality: $]x[\, \geq x$.
13. $i:=1$, $\varepsilon=0$
14. We generate a random number ξ evenly distributed on $[0,1]$ interval
15. $\varepsilon := \varepsilon + I_{\{\xi \leq \nu[k]/n\}}$
16. $i:=i+1$
17. If $i \leq n$, then 14
18. $\varepsilon 2 := \varepsilon 2 + b[k]\varepsilon$
19. $k:=k+1$
20. if $k \leq N$, then 12
21. $\varepsilon := \varepsilon 1 + \varepsilon 2$
22. End of procedure.

Let's consider the example of generation of self-similar Levy process with cumulant $\psi(\theta) = i\theta m - \frac{\sigma^2}{2}\theta^2 + \int_{-\infty}^{\infty} \left(\exp(i\theta x) - 1 - i\theta x I_{(-1,1)}(x)\right)\nu(dx)$ with Levy measure: $\nu(dx) = \frac{C_1}{x^{1+\alpha}} I_{(0,+\infty)}(x)dx + \frac{C_2}{|x|^{1+\alpha}} I_{(-\infty,0)}(x)dx$. We approximate this process by the process with cumulant $\psi_\delta(\theta) = i\theta\bar{m} - \frac{\bar{\sigma}^2}{2}\theta^2 + \int_{|x|\geq\delta} \left(\exp(i\theta x)-1\right)\nu(dx)$ and the same Levy measure. Figures 3 – 6 present trajectories of stochastic processes with various indices of stability.

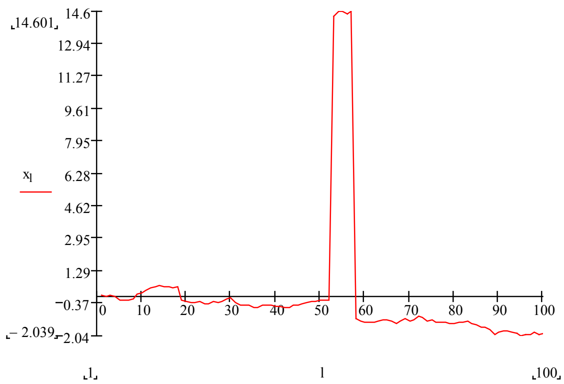

Figure 3. Trajectory of stochastic process with $\alpha = 0.5, c1 = 0.005, c2 = c1, \sigma = 0.1, m = 0.01$.
Simulation interval [0,1], 100 values

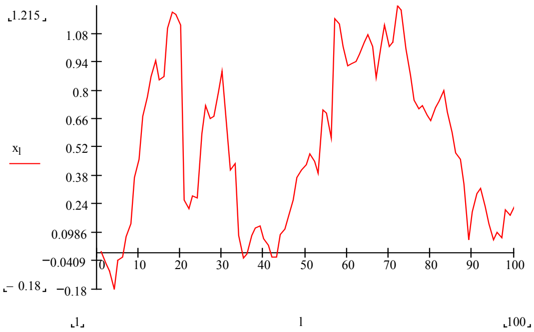

Figure 4. Trajectory of stochastic process with $\alpha = 1, c1 = 0.01,\ c2 = 0.5c1, \sigma = 0.1, m = 0.01$. Simulation interval [0,1], 100 values

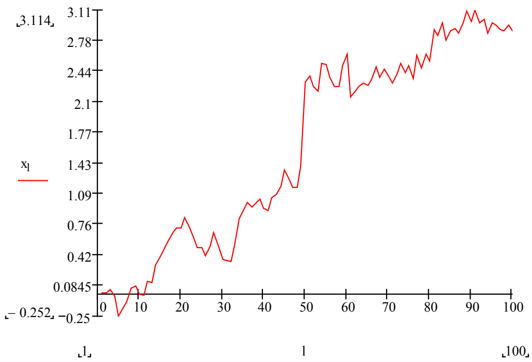

Figure 5. Trajectory of random stochastic process with $\alpha = 1.5, c1 = 0.01,\ c2 = 0.5c1, \sigma = 0.1, m = 0.01$. Simulation interval [0,1], 100 values

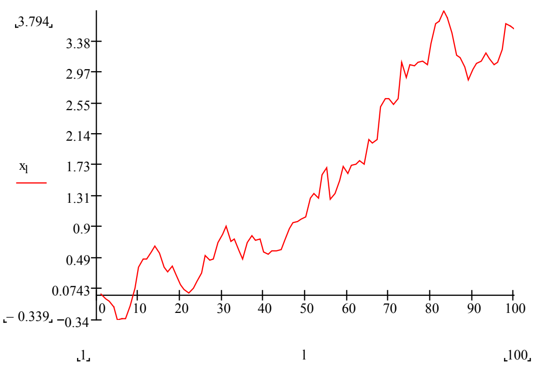

Figure 6. Trajectory of stochastic process with $\alpha = 2, c1 = 0.01,\ c2 = 0.5c1, \sigma = 0.1, m = 0.01$. Simulation interval [0,1], 100 values

As the index of stability increases the curve becomes more broken, and the number of big jumps decreases.

D. Simulation of Levy processes with use of subordination

In this section we consider the subordination of Wiener process.That is, we consider the process

$$X_t = W_{T_t}, \qquad (25)$$

где T is Levy subordinator.

As it was mentioned in the first chapter, the process T is the process with bounded variation and its cumulant:

$$\psi_T(\theta) = i\theta m + \int_0^{\infty} \left(\exp(i\theta x) - 1\right)\nu(dx),$$

$$\int_{-\infty}^{0} \nu(dx) = 0, \int_0^{\infty} (x \wedge 1)\nu(dx) < \infty$$

Hence, the numerical algorithm of subordinator generation has the following form.

Algorithm 5 of subordinator generation

Procedure $SUB(\mu, a, M, h, m, \zeta, \varepsilon)$

1. $k := 1, \varepsilon := hm$
2. $n := \left]\frac{\mu(k)^2}{\zeta}\right[$, where $]x[$ is the minimal integer that meets inequality: $]x[\geq x$.
3. $i := 1$
4. We generate a random number ξ evenly distributed on $[0,1]$ interval
5. $\varepsilon := \varepsilon + I_{\{\xi \leq \mu[k]/n\}} a[k]$
6. $i := i + 1$
7. If $i \leq n$, then 4
8. $k := k + 1$
9. If $k \leq M$, then 2
10. End of Procedure.

Let's consider the increment of the process X - $\varepsilon = W_{T_{ih}} - W_{T_{(i-1)h}}$. The increment on distribution is equal to $\sqrt{T_h}\xi$, where ξ is a standard normal variable. Hence, the numerical algorithm of generation of subordinated Wiener process has the form.

Algorithm 6 of generation of subordinated Wiener process

Procedure $SUBW(\mu, a, M, h, m, \zeta, \varepsilon)$

1. $SUB(\mu, a, M, h, m, \zeta, \varepsilon 1)$
2. Generation of a normal standard random variable ξ
3. $\varepsilon := \sqrt{\varepsilon 1}\xi$
4. End of procedure.

Let's consider the example, when Poisson process is the subordinator. The process trajectory is given on fig. 7.

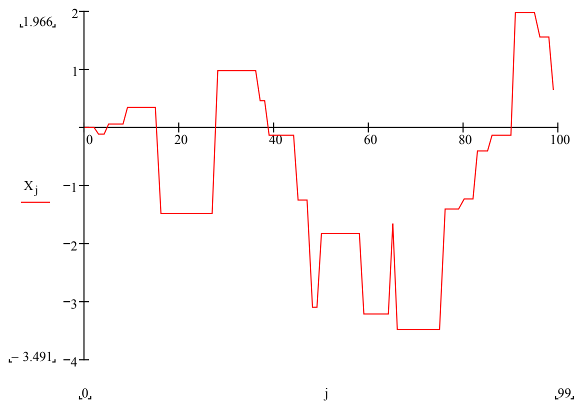

Figure 7. Trajectory of subordinated Wiener process. Subordinator is the Wiener process, with intensity $\lambda = 30$. Simulation interval [0,1], 100 values

The procedure in combination with procedure $SUB(\mu, a, M, h, m, \zeta, \varepsilon)$ is universal enough, however, there are situations when it is possible to construct a more simple algorithm. The example of such situation are Levy processes on the basis of distribution of hyperbolic type. In particular, on the basis of Gaussian/inverse Gaussian distribution. The density of distribution is defined by formula 10.

When constructing Levy process on the basis of Gaussian/inverse Gaussian distribution we will use the following fact [9]. Let's consider the subordinator $T_t = \inf\left\{s \geq 0 : W_s + \sqrt{a s} \geq \sqrt{b t}\right\}$, where $a = \alpha^2 - \beta^2, b = \delta^2$.

Then the process $SUBW(\mu, a, M, h, m, \zeta, \varepsilon)$

$$X_t = W_{T_t} + \mu t + \beta T_t \tag{26}$$

will be the Levy process on the basis of Gaussian/inverse Gaussian distribution From 26 it follows, that the increment

$$\varepsilon = X_{ih} - X_{(i-1)h} = \mu h + \sqrt{T_h}\xi + \beta T_h \ , \tag{27}$$

where $T_h = h \inf\left\{j : W_{jh} + \sqrt{a}\, jh \geq \sqrt{b h}\right\}$.

Equations (27) allow to propose the following numerical method of process simulation.

Algorithm 7 generation of Gaussian/inverse Gaussian distribution

Procedure $GIG(m, a, b, h, \beta, \varepsilon)$

1. $W := 0, j := 0$
2. We generate ξ
3. $W := W + \xi$
5. $j := j + 1$
6. If $\sqrt{h}W + \sqrt{a}\, jh < \sqrt{b h}$ then 2
7. We generate ξ
8. $\varepsilon := mh + \sqrt{jh}\xi + \beta jh$
9. End of procedure.

Figure 8 gives the trajectory of Levy process on the basis of Gaussian/inverse Gaussian distribution.

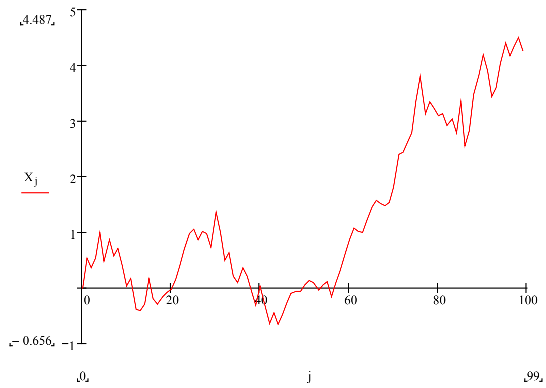

Figure 8. Trajectory of Levy process on the basis of Gaussian/inverse Gaussian distribution ($a = 1, b = 2, \beta = 0.1, m = 0.01$). Simulation interval [0,1], 100 values

VII. CONCLUSION

In the paper, we present new mathematical models of pure jump stochastic processes Levy type, based fact of time substitution. Also the paper proposes a few new simulation algorithms, which are faster and less complex relative to known.

ACKNOWLEDGEMENT

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