Limit Cycles in High-Resolution Quantized Feedback Systems

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***Abstract*—In this paper, the existence of limit cycles in high-resolution quantized feedback systems is examined. It is well known that the relay and the quantized feedback systems exhibit self-oscillations, due to their switching nature. However, the quantizer is a more general nonlinearity as compared to the relay, due to its switching at multiple discrete levels. An extension of periodic switching conditions uncovers the existence of self-oscillations in some systems under high quantization resolution. Multiple limit cycle solutions of switching instants and periods have been found, depending on the initial states of the system. Further analysis on the stability of the limit cycle via the Jacobian of the Poincaré map reveals numerical bounds on the quantization step size for a stable limit cycle. Analytical results on the existence of limit cycles in first and second order systems are also presented.**

Keywords-Quantized Systems, Limit Cycles, Stability.

I. INTRODUCTION

As early as 1956, Kalman studied the effect of quantization in a sampled data system and pointed out that the feedback system with a quantized controller would exhibit limit cycles and chaotic behavior [1-2]. Since then, various methods to eliminate limit cycles have been proposed in SISO and MIMO quantized feedback systems such as increasing the quantization resolution, dithering the quantizer with a DC signal and stabilizing controllers design [3-9]. As compared to the other methods, the most direct method, which is to increase the quantizer resolution, will be examined in this paper.

In past literature, a standard assumption is that the quantizer parameters are fixed in advance and cannot be changed. However, in a real-life system like the digital camera, the resolution can be easily adjusted in real time [10]. Hence, we adopt the approach that the quantizer resolution can be adjusted. In this paper, the problem structure we examine is the hybrid system, which is a continuous-time system with a uniform quantizer in feedback. The recent paper by Brockett and Liberzon shows that if a linear system can be stabilized by a linear feedback law, then it can also be globally asymptotically stabilized by a hybrid quantized feedback control policy [11].

Under high quantizer resolution, the uniform quantizer resembles a linear gain with many minute switches. Hence, if the continuous-time system is stable under negative closed loop feedback, the hybrid system is indeed expected to stabilise. However, limit cycles have been found to exist under certain conditions with high quantizer resolution. There exist literature on the conditions required for limit cycles [12-13] but the problem under high resolution has not been examined, to the best of our knowledge. Thus, there is a need to study the behavior of the system under high resolution in greater depth. For the evaluation of the limit-cycle properties of the hybrid system, the inverse-free Newton's method is used [15]. As the inverse Jacobian for the hybrid system does not exist in many cases, the conventional Newton's method cannot be applied.

Multiple solutions of the switching instants and period can be obtained with the inverse-free Newton's method, depending on the initial states of the system. Due to multiple discrete levels in the quantizer, it provides an additional degree of freedom for the limit-cycle characteristics. For instance, both a 1-step limit cycle and a 2-step limit cycle can be reached with different initial conditions in a hybrid system with a 40-step quantizer and the switching instants and the periods of each limit cycle can differ. Thus, the limit cycle solution is non-unique, unlike the relay (1-step quantizer) feedback system. This additional degree of freedom can be reduced by fixing the number of levels expected in a limit cycle. If so, we are able to identify the limit cycle solution through the necessary conditions required. Specifically, Jacobian conditions for locally stable limit cycles are analyzed to obtain numerical bounds on quantization resolution. In some cases, the limit cycle is stable for high quantization resolution in the range of one to two thousands.

This paper is structured as follows. The problem formulation is discussed in Section II. In Section III, the analysis on the necessary and sufficient conditions for the existence and stability of limit cycles are shown. Further results on the identification of the bounds on the quantization step size are presented in Section III-C. Conclusions are given in Section IV.

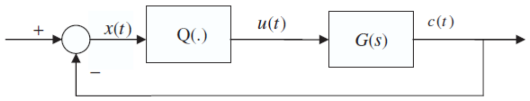

Figure 1: Quantized feedback system.

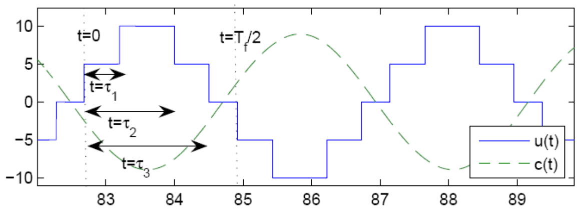

Figure 2: 5-level limit cycle.

II. PROBLEM FORMULATION

Consider the quantized feedback system with a finite limit midtread quantizer $Q(x)$, as shown in Fig. 1. The linear system, $G(s)$, is assumed to have a state space description given by

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) \\ c(t) &= Cz(t) \end{aligned} \qquad (1)$$

with

$$\begin{aligned} u &= j\Delta,\ (j-0.5)\Delta < x \le (j+0.5)\Delta \\ |u| &\le M \end{aligned}$$

where $j \in \mathbf{Z}$, $u, c \in \mathbf{R}$ are the input and output, respectively, $z \in \mathbf{R}^{m\times 1}$ is the state vector, $A \in \mathbf{R}^{m\times m}$ is Hurwitz and assumed to be non-singular, $B \in \mathbf{R}^{m\times 1}$, $C \in \mathbf{R}^{1\times m}$, $M, \Delta \in \mathbf{R}$ are the saturation limit and step size of the quantizer respectively.

Assumption 1: It is assumed that the system (1) and (2) is stable under unit negative feedback.

Given a positive real number Δ and $M \in \mathbb{Z}$, we define the quantizer, $Q_\Delta(x)$ by the formula

$$Q_\Delta(x) = \begin{cases} M, & if\ x > M + 0.5\Delta \\ -M, & if\ x \le -M - 0.5\Delta \\ \lfloor \frac{x}{\Delta} + 0.5 \rfloor \Delta, & if\ -M - 0.5\Delta < x \le M + 0.5\Delta \end{cases}$$

where M denotes the saturation limits of the quantizer, Δ denotes the step size and $\Delta = 2M/(k-1)$ for k being the number of quantization levels.

The quantizer $Q_\Delta(x)$ takes in a continuous input $x \in \mathbb{R}$ and outputs a discrete value u. Discontinuities in u occur at the boundaries of the quantization intervals. When $-M - 0.5\Delta < x \le M + 0.5\Delta$, the quantization error satisfies the bound $|x - u| \le 0.5\Delta$.

When $|x| > M + 0.5\Delta$, the finite limit quantizer saturates at a value $u = M$. In this case, the quantizer error will be at least 0.5Δ for the uniform quantizer being studied. If the quantizer does not saturate, the quantization error can be reduced with a smaller Δ. Given a stable closed loop system in the absence of the quantized nonlinearity, the output would have converged to zero. In a similar sense, if we consider the quantized feedback system where the quantization error can be continually reduced by decreasing Δ, the output may converge to the origin asymptotically. The presence of quantization errors due to saturation and deterioration of performance near the equlibrium are manifested in the existence of two nested invariant regions such that all trajectories of the quantized system starting in the bigger region (at saturation limit) approach the smaller one (near the equilibrium) when Δ is decreased. The invariant region consists of the levels sets of a Lyapunov function. When the state trajectory enters a level set, it does not leave. The size of level set is related to Δ. This behaviour corresponds to limit cycling with different amplitudes in time domain where the amplitude is related to Δ.

In our study, we refer to the boundaries as switching planes and denote the time instants where periodic switching occur as $(0, \tau_1, \tau_2, \ldots, \tau_{2k'-1}, T/2)$ where $k' = 0.5(k-1)$, k is the number of quantization levels and $T/2$ being the half-period of the symmetrical limit cycle. Consider an example of a limit cycle for a 5-level quantizer and a plant with transfer function $G(s) = \frac{6.5}{s^3+s^2+2s+4}$. For the 5-level quantizer ($k = 5, \Delta = 5$), the switching planes occur at $(-1.5\Delta, -0.5\Delta, 0.5\Delta, 1.5\Delta)$ and the switching time instants, $(0, \tau_1, \tau_2, \tau_3, T/2)$are as t in Figure 2, where $t = 0$ is relative to a positive switching edge.

In frequency domain, the open loop transfer function of the linear system is $G(s) = e^{-sL}C(sI - A)^{-1}B$ where $G(s)$ is strictly proper and satisfies $\lim_{s\to\infty} G(s) = 0$.

Assuming that a k-level symmetric limit cycle exists, the output of the quantizer is given by

$$u(t) = \frac{1}{2\pi}\Delta\left(\left(\sum_{n=-\infty}^{\infty} \frac{1-e^{-in\omega\tau_{2k'-1}}}{n} e^{in\omega\tau} + \left(\sum_{n=-\infty}^{\infty} \frac{e^{-in\omega\tau_1}(1-e^{-in\omega\tau_2})}{n} e^{in\omega t}\right.\right.\right.$$

$$\left.\left.\left. + \quad \ldots \quad + \left(\sum_{n=-\infty}^{\infty} \frac{e^{-in\omega\tau_{k'-1}}(1-e^{-in\omega\tau_{k'}})}{n} e^{in\omega t}\right)\right.\right.\right).$$

The corresponding plant output,

$$\begin{aligned} c(t) = \quad & \frac{2\Delta}{n\pi}\sum_{n=odd}^{\infty}\Big(\mathrm{Im}\{G(jn\omega)e^{jn\omega t}\} - \mathrm{Im}\{G(jn\omega)e^{jn\omega t}\} \\ & \cos(n\theta_{2k'-1}) + \mathrm{Re}\{G(jn\omega)e^{jn\omega t}\}\sin(n\theta_{2k'-1}) + \ldots + \\ & \mathrm{Im}\{G(jn\omega)e^{jn\omega t}\}\cos(n\theta_{nk'-1}) - \mathrm{Re}\{G(jn\omega)e^{jn\omega t}\} \\ & \sin(n\theta_{nk'-1}) - \mathrm{Im}\{G(jn\omega)e^{jn\omega t}\}\cos(n(\theta_{k'} + \theta_{k'-1})) \\ & + \mathrm{Re}\{G(jn\omega)e^{jn\omega t}\}\sin(n(\theta_{k'} + \theta_{k'-1})). \end{aligned}$$

The A-locus [19] which is essentially the phase portrait of $c(t)$ at $t = 0$ for different values of ω can be written as

$$\begin{aligned} \Lambda^*(\theta_1, \theta_2, \ldots, \theta_{2k'-1}, \omega) = \quad & \Lambda(0,\omega) - \Lambda(\theta_{2k'-1},\omega) + \Delta(\Lambda(\theta_1,\omega) \\ & -\Lambda(\theta_1 + \theta_2, \omega)) + \ldots + \Delta(\Lambda(\theta_{k'-1},\omega) \\ & -\Lambda(\theta_{k'-1} + \theta_{k'}, \omega)) \end{aligned}$$

where $k' = 0.5(k-1)$, $\theta_i = \omega\tau_i$ and $i = 1, 2, ..., k'$.

The solution for k-level limit cycle can be found if the set $\theta_1, \theta_2, ..., \theta_{2k'-1}, \omega$ is known. However, this parameter set to achieve a k-level limit cycle is usually not given *apriori*. In addition, the k-level limit cycle may not exist. It might be possible to obtain the parameters required by a trial and error process using the A-loci. However, this would not be manageable for a large k. Another continuous time representation of the system, the state space description in time domain will be used in our study.

In time domain, the state trajectory of \$z(t)\$ for a \$k\$-level limit cycle can be expressed as

$$z(t) = \begin{cases} e^{A(t)}z(0) + \int_0^t {}^{A(t-\tau)}B\Delta d\tau, 0 < t < \tau_1 \\ e^{A(t-\tau_1)}z(\tau_1) + \int_{\tau_1}^t {}^{A(t-\tau)}B2\Delta d\tau, \tau_1 < t < \tau_2 \\ \vdots \\ e^{A(t-\tau_i)}z(\tau_i) + \int_{\tau_i}^t {}^{A(t-\tau)}B(i+1)\Delta d\tau, \tau_i < t < \tau_{i+1} \\ \vdots \\ e^{A(t-\tau_j)}z(\tau_j) + \int_{\tau_j}^t {}^{A(t-\tau)}B(j-2k'+1)\Delta d\tau \\ , \tau_j < t < \tau_{j+1} \end{cases}$$

where τ_is and τ_js are the time instants when the state trajectory traverses the switching planes for $i = 2, 3, ..., k'-1$ and $j = k', k'+1, ..., 2k'-2$.

Specifically, the necessary conditions where limit cycles of various levels can exist and their stability will be examined in continuous time using the state space representation. In order to maintain a constant saturation limit, we preset the quantization step size Δ as $2M/(k-1)$and refer to the quantization resolution as an inverse function of the step size. Thus, it is expected that as the number of quantization level k increases, the quantization step size Δ will decrease and result in a higher quantization resolution. In the next section, the necessary conditions for the existence of multiple odd limit cycles, the stability of the solution, the behavior at convergence and results on first and second order systems will be discussed.

III. ANALYSIS

A. Odd Limit Cycles

The necessary conditions for odd limit cycles will be examined in this section. The derivation of the numerical solution will be presented.

Proposition 1: Consider the quantized feedback system as given by (1) and (2). Assume that there exists a symmetric k-level periodic solution with switching times $(\tau_1, \tau_2, \ldots \tau_{2k'-1})$ and period T where $k' = 0.5(k-1)$. An extension of the necessary conditions in [14] lead to the following.

$$\begin{aligned} Cz(\tau_i) &= Ce^{A(\tau_j-\tau_i)}z(\tau_i) + C\int_{\tau_i}^{\tau_j} e^{A(t-\tau)}Bw\Delta d\tau \\ &= -(w-0.5)\Delta \\ Cz(T/2) &= Ce^{A(T/2-\tau_{2k'-1})}z(\tau_{2k'-1}) \\ &= 0.5\Delta \end{aligned} \quad (2)$$

where $\tau_0 = 0$, $i = j-1$, $j = 1, 2, \ldots, 2k'-1$, $w = k - |s|$ and $s = -k+1, -k+2, \ldots, k-2, k-1$.

$$\begin{aligned} \frac{\partial Cz(t)}{\partial t}|_{t=\tau_p} &< 0 \\ \frac{\partial Cz(t)}{\partial t}|_{t=\tau_q} &> 0 \\ \frac{\partial Cz(t)}{\partial t}|_{t=\tau_{T/2}} &> 0 \end{aligned} \quad (3)$$

where $p = 1, 2, \ldots, k'-1$ and $q = k', k'+1, \ldots, 2k'-1$. Furthermore, the periodic solution is obtained with the initial condition

$$z(0) = -z(T/2) \quad (4)$$

Numerical methods are required to find the limit cycle solution $(\tau_1, \tau_2, \ldots, \tau_{2k'-1}, T/2)$ which satisfies (3), due to more unknowns than conditions. Usually, the conventional Newton Raphson method can be used. However due to the large number of zeros present in the Jacobian matrix, its inverse usually does not exist for computations where k is large. Thus, an alternative method has to be used, which is the inverse-free Newton's Method in [15]. The use of the method is briefly described below.

Rearranging the equations in (3),

$$\begin{aligned} Cz(\tau_i) + (w-0.5)\Delta &= 0 \\ Cz(T/2) - 0.5\Delta &= 0 \end{aligned} \quad (5)$$

The solution of (6) is $(\tau_1, \tau_2, \ldots, \tau_{2k'-1}, T/2, z_m^0)$ where z_m^0 is the state at the m-th switching point where $m = 0, 1, 2, \ldots, 2k'-1$.

Denoting the system of nonlinear equations (6) by F,

$$F(\tau_1, \tau_2, \ldots, \tau_{2k'-1}, T, z_m^0) = 0 \quad (6)$$

Let $b = [\tau_1; \tau_2; \ldots; \tau_{2k'-1}; T, z_m^0]$ and

$$F(b) = 0 \quad (7)$$

By the inverse-free Newton's Method, let $F_1 = \frac{1}{2}F^T F$and the Jacobian is $J_1 = F^T J$ where J is the Jacobian $\partial F/\partial b$. By the updating algorithm, $b_n = b_{n-1} - F_1\frac{J_1}{||J_1||^2}$, The b_n can be iteratively updated till the error $b_n - b_{n-1}$ converges to zero. The effectiveness of this method is verified by the following example.

Example 1: Consider the plant with transfer function, $G(s) = \frac{6.5}{s^3+4s^2+2s+1}$ and a 3-level quantizer where $M = \Delta = 5$ in closed loop feedback. In the absence of the quantizer, the closed loop poles are at -3.9719, $-0.0141 \pm 1.3741i$. With the 3-level quantizer, a limit cycle of 1 step with $(\tau_1, T) = (1.5302, 4.4646)$ is obtained, as shown in Figure 3.

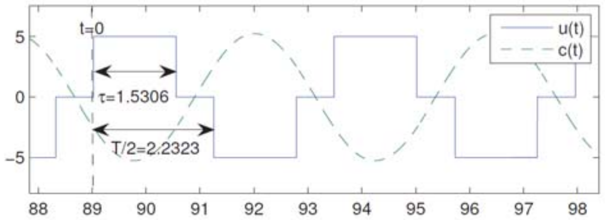

Figure 3: 3-level limit cycle.

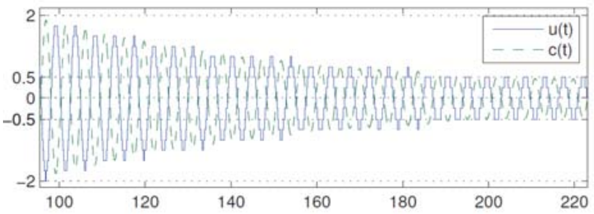

Figure 4: 41 level limit cycle with $\Delta = 0.25$

By the inverse-free Newton's method, after 1500 iterations,

$$J = \begin{bmatrix} -14.0990 & 0 & 7.0259 & 1.5717 \\ -23.4021 & 9.9244 & 7.8565 & 1.9069 \\ -1.4069 & -0.7110 & 1.0406 & 0.0353 \\ 1.7068 & -0.7961 & -0.3640 & 0.8996 \end{bmatrix}$$

$$b_{1500} - b_{1499} = 10^{-4} \begin{bmatrix} -0.1115 \\ -0.0337 \\ -0.0392 \\ -0.0090 \end{bmatrix}$$

and

$$(\tau_1, T, z_2(0), z_3(0)) = (1.5306, 4.4646, -.0117, -0.4832).$$

Note that in the Jacobian of the solution, the first element of the first row and the second element from the left of the second row correspond to the gradient at the switching instants. The gradient conditions in (4) have been checked to ensure that the solution satisfies all the necessary conditions in (3) and (4).

B. Stability of Limit Cycles

In this section, the local stability of the limit cycle with the computed switching times and period, will be checked as shown in the Proposition below. The stability of the limit cycle is analyzed by studying the effect of perturbations at each switching time instant. For example, to check the stability of the limit cycle about a certain point at$t = \tau_i$, we apply a perturbation at $t = \tau_i$ and check its effects at $t = \tau_i + T/2$. For a *k*'-step odd symmetric limit cycle, the switching time instants are$(0, \tau_1, \tau_2, \ldots, \tau_{2k'-1}, T/2)$. It will be shown in the proof that if these *2k*' switching instants are studied, the eigenvalues of *2k*' Jacobians resulting from the Poincaré maps originating from each switching time instant, is required to be inside the unit disk. A further examination reveals that the Jacobians would have the same eigenvalues and it suffices to examine the eigenvalue of one Jacobian. We choose the Jacobian of the Poincaré map originating from $t = 0$, similar to [16-18].

Proposition 2 (Local Stability): Consider the system with closed loop quantized feedback in Figure 1. Assume that there is a *k*-level symmetric periodic solution. Let a_i be the state of the system when it traverses each switching plane and d_w be the corresponding quantizer output value. The corresponding Jacobian of the Poincaré map is given by

$$W = \prod_{i=1}^{k-1} W_i \tag{8}$$

where

$$W_i = (I - \frac{\nu_i C}{C\nu_i})\Phi_i \tag{9}$$

$\Phi_i = e^{A(\tau_i - \tau_{i-1})}$, $\tau_0 = 0$ $\nu_i = Aa_i + Bd_w$, $\quad w = k - |s|$ and $s = -k+1, -k+2, \ldots, k-2, k-1$. The limit cycle is locally stable if and only if all eigenvalues of *W* are inside the unit disk.

Proof: Refer to Appendix.

Remark 1: The local stability of each traversal point of the limit cycle is checked. Note that without the computed switching times and period, the Jacobian *W* cannot be evaluated.

Remark 2: One of the eigenvalues of the Jacobian is 0 as *C* is a left eigenvector of $(I - \frac{\nu_i C}{C\nu_i})$.

Consider the same plant in Example 1 with a 5-level quantizer with $\Delta = 2.5$. A 2-step limit cycle solution LC_1:

$$(\tau_1, \tau_2, \tau_3, T, z_1(0), z_2(0), z_3(0)) = (0.5347; 1.2953; 1.8123; 4.4532, 0.1918, 0.8974, -0.3020)$$

is locally stable by Proposition 2. The eigenvalues of (9) are $0.0101606, 0.3296, 0$ which are in the unit disk. We also found that the quantizer with a step size of $\Delta = 0.0005$produced a 2-step limit cycle

$$(\tau_1, \tau_2, \tau_3, T, z_1(0), z_2(0), z_3(0)) = (0.533, 1.298, 1.814, 4.454, 0.00004, 0.00017, -0.00006).$$

By Proposition 2, we find that the eigenvalues are $0.0102, 0.3304, 0$, which are within the unit disk and the limit cycle is locally stable. This has been verified in simulation using a 41-level quantizer. For $\Delta = 5/20 = 0.25$, the quantizer output $u(t)$ converged to a 2-step limit cycle of amplitude $2 \times 0.25 = 0.5$, as shown in Figure 4. If we further decrease the step size Δ to 0.0005, the quantizer output again converged to a 2-step limit cycle of amplitude $2 \times 0.0005 = 0.001$, as shown in Figure 5. As the quantization step size decreased from 0.25 to 0.0005, the quantizer output amplitude has decreased accordingly but the 2 step limit cycle remains at steady state.

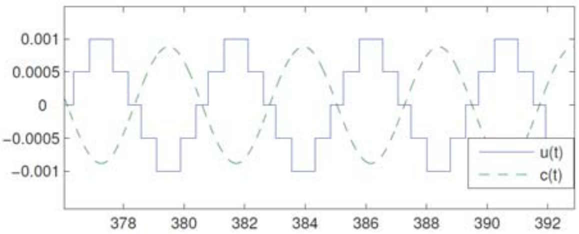

Figure 5: 41 level limit cycle with $\Delta = 0.0005$

C. Special Cases

First order plants without delay

For first order plants without delay, it is generally well known that first order plants do not self-oscillate under relay feedback. It can be checked that a first order non-delayed plant does not self-oscillate when placed in closed loop with a quantizer by applying the conditions in (3).

First order plants with delay

For first order plants with delay, $W = 0$ and the limit cycle is always locally stable. Take for instance, in first order systems with a 3-level quantizer and delay where $0 < L < \tau$ and $0 < \tau < T/2$, it is required that $T/2 < L + \tau < T$ and $T > 2L + \tau$. This is shown as follows.

Consider two cases: $0 < L + \tau < T/2$ and $T/2 < L + \tau < T$.

Case 1: $0 < L + \tau < T/2$

$$\begin{aligned} Cz(\tau) &= Ce^{A\tau}z(0) - C(I - e^{A(\tau-L)})A^{-1}B\Delta \\ &= -0.5e^{A\tau}\Delta - C(I - e^{A(\tau-L)})A^{-1}B\Delta \end{aligned} \quad (10)$$

For a limit cycle to exist, $Cz(\tau) = -0.5\Delta$. As the right hand side of (11) is positive, no limit cycle is possible for $0 < L + \tau < T/2$.

Case 2: $T/2 < L + \tau < T$

$$Cz(\tau) = Ce^{A\tau}z(0) - Ce^{A(T/2-L)}(e^{A(\tau+L-T/2)} - I)A^{-1}B\Delta \quad (11)$$

$$= -(0.5e^{A\tau} + Ce^{A(T/2-L)}(e^{A(\tau+L-T/2)} - I)A^{-1}B)\Delta \quad (12)$$

The right hand side in (13) is negative and the switching condition $Cz(\tau) = -0.5\Delta$ can be satisfied. Next, the switching condition at $t = T/2$ is examined.

$$Cz(T/2) = Ce^{A\tau}z(0) + Ce^{A(L)}(e^{A(T-2L-\tau)} - I)A^{-1}B\Delta \quad (13)$$

If the switching condition at $t = T/2$ is satisfied,

$$\begin{aligned} 0.5(I + e^{A\tau}) &= Ce^{A(L)}(e^{A(T-2L-\tau)} - I)A^{-1}B \\ e^{A(T-2L-\tau)} - I &< 0 \\ T &> 2L + \tau \end{aligned} \quad (14)$$

Thus, for first order plants with delay, a limit cycle cannot exist for $0 < L + \tau < T/2$ and $T < 2L + \tau$.

Second order plants

For second order plants with state space representation, $A = [0\ 1;\ -\lambda_1\lambda_2\ (\lambda_1 + \lambda_2)]$, $B = [0\ 1]^T$ and $C = [c\ 0]$ where $\lambda_1 < \lambda_2 < 0$ are the roots of the plant, limit cycles may not always exist. For second order plants without delay, when the state trajectory, $z(t)$ enters the deadzone region, it tends to the origin exponentially and stays at the origin. Thus, no limit cycle exist. For second order systems with delay, a limit cycle may exist if the necessary conditions in (3) and (4) are satisfied.

Section III-B presents that the limit cycle if exist, is locally stable if the eigenvalue of the Jacobian (9) in Proposition 2 lies in the unit circle. By analyzing (9) carefully, the range of Δ for the existence of limit cycles can be determined. The result is captured in the following proposition for a second order delayed plant with a 3-level quantizer.

Proposition 3: For a second order plant in negative feedback with a 3-level quantizer, the limit cycle with solution set (τ, T, L) is locally stable if and only if

$$\begin{aligned} \Delta &> \max\{\Delta_1\}, \\ \max\{\Delta_2\} < \Delta &< \min\{\Delta_3\}, \\ \Delta &> \max\{\Delta_3, \Delta_4\} \end{aligned} \quad (15)$$

or

$$\begin{aligned} \Delta &< \min\{\Delta_1\}, \\ \max\{\Delta_4\} < \Delta &< \min\{\Delta_3\}, \\ \Delta &> \max\{\Delta_3\}, \end{aligned} \quad (16)$$

where Δ_1, Δ_2, Δ_3, and Δ_4 are defined in Table I and $\lambda_1\lambda_2(e^{\lambda_2 L+0.5\lambda_1 T+2\lambda_2\tau} - e^{\lambda_1 L+0.5\lambda_2 T+2\lambda_1\tau}) + 2e^{0.5(\lambda_1+\lambda_2)T}(e^{\lambda_1\tau} - e^{\lambda_2\tau}) > 0$

As shown in the Proposition 3, at quantization step size $\Delta > \max\{\Delta_1\ \Delta_3, \Delta_4\}$, the limit cycle is locally stable. By reducing the quantization step size, the limit cycle becomes unstable and may disappear. Given that the quantization step size $\Delta \geq 0$,the limit cycle may exist if $\min\{\Delta_3\} \geq 0$.

Consider an example, $A = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 6.5 & 0 \end{bmatrix}$, $L = 1$. For this plant at $\Delta = 0.0005$, the limit cycle $(\tau, T, z_1(0), z_2(0)) = (2.819, 3.002, -0.0000387, -0.0004985)$ exists. By Proposition 3, the limit cycle $(\tau, T) = (2.819, 3.002)$ is stable for $\Delta < 0.0001849$, $\Delta > 0.0132733$ for negative eigenvalues. For positive eigenvalues, the limit cycle is stable for $0.000233131 < \Delta < 0.00849123$, $0.00849123 < \Delta < 0.0132733$. This further confirms the existence of LC at $\Delta = 0.0005$.

In this section, the stability of the limit cycle has been studied. One of the parameters that affect the stability of the limit cycle solution is the quantization step size Δ. The limits on Δ was identified by evaluating the magnitude of the eigenvalues of the Jacobian W for a range of Δ. In a

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particular example, it was found that the quantizer output converged to a 2-step limit cycle of a small amplitude at small quantization step sizes. Note that by increasing the quantization resolution, a 2-step limit cycle with a small amplitude was obtained. The special cases examined, reveals the conditions required for limit cycles to exist. For a second order plant with a 3-level quantizer, the bounds of the quantization step size for stable limit cycles have been identified.

IV. CONCLUSIONS

In this paper, the necessary conditions for the existence of limit cycles with various levels and their stability have been examined in continuous time. A study of the local stability of the limit cycles was performed by analyzing the eigenvalues of the Jacobian of the Poincaré map for each switching instant. It was shown that the Jacobians for each switching instant have the same eigenvalues and it suffices to analyze only one Jacobian for local stability. At high quantization resolution, the system with the uniform quantizer may converge exponentially to a limit cycle whose amplitude is related to Δ. The stability of the limit cycle can be identified by evaluating the magnitude of the eigenvalues of the Jacobian W of the Poincaré map. From the local stability result, a bound on the step size was identified and it was found that limit cycles can still exist under high quantization resolution. In a particular example, it was shown that for a small quantization step size, the output converged exponentially to 1 2-step limit cycle that is stable for $\Delta \geq 0$. Further results on the existence of limit cycles in first order systems were also presented.

V. APPENDIX

Proof of Proposition 2

Consider the trajectory resulting from the perturbed initial condition $z(0) = a_0 + \delta a_0$. The perturbation is chosen such that it satisfies the switching condition

$$C(a_0 + \delta a_0) = -0.5\Delta. \tag{17}$$

The perturbed solution is

$$z(t) = e^{At}(a_0 + \delta a_0) + \int_0^t e^{A(t-s)} Bu(s)ds.$$

Assume that the solution reaches the first switching plane at time $\tau_1 + \delta\tau_1$. Hence,

$$\begin{aligned} z(\tau_1 + \delta\tau_1) &= z(\tau_1) + \Phi_1\delta a_0 + (Az(\tau_1) + B\Delta)\delta\tau_1 + O(\delta^2) \\ &= a_1 + \Phi_1\delta a_0 + \nu_1\delta\tau_1 + O(\delta^2) \end{aligned} \tag{18}$$

where $z(\tau_1) = a_1$, $\Phi_1 = e^{A\tau_1}$ and $\nu_1 = Aa_1 + Bd_0$. For $Ca_1 = Cz(\tau_1 + \delta\tau_1)$, we get $\delta\tau_1 = -\frac{C\Phi_1}{C\nu_1}\delta a_0 + O(\delta^2)$. Inserting this in (19) gives

$$z(\tau_1 + \delta\tau_1) = a_1 + (I - \frac{\nu_1 C}{C\nu_1})\Phi_1\delta a_0 + O(\delta^2) \tag{19}$$

The perturbation at time $\tau_1 + \delta\tau_1$ is thus given by $\delta a_1 = (I - \frac{\nu_1 C}{C\nu_1})\Phi_1\delta a_0 + O(\delta^2)$.

In the same way, we can study how the perturbation δa_1 of a_1 affects the solution at time $\tau_1 + \delta\tau_1 + \tau_2 + \delta\tau_2$. We get

$$\begin{aligned} z(\tau_1 + \delta\tau_1 + \tau_2 + \delta\tau_2) &= a_2 + (I - \frac{\nu_2 C}{C\nu_2})\Phi_2\delta a_1 + O(\delta^2) \\ &= a_2 + (I - \frac{\nu_2 C}{C\nu_2})\Phi_2 1(I - \frac{\nu_1 C}{C\nu_1})\Phi_1\delta a_0 + O(\delta^2). \end{aligned} \tag{20}$$

We follow through the same analysis till time $\tau_1 + \delta\tau_1 + \tau_2 + \delta\tau_2 + \cdots + T/2 + \delta T/2$. Finally,

$$\begin{aligned} &z(\tau_1 + \delta\tau_1 + \tau_2 + \delta\tau_2 + \cdots + T/2 + \delta T/2) \\ &\quad = -a_0 + \prod_{i=1}^{k-1}(I - \frac{\nu_i C}{C\nu_i})\Phi_i\delta a_0 + O(\delta^2). \end{aligned} \tag{21}$$

The Jacobian of the Poincaré map is given by (9).

Next, consider the trajectory resulting from the perturbed initial condition $z(\tau_1) = a_1 + \delta a_1$. We follow through the same analysis and the Jacobian of the Poincaré map is $W = (\prod_{i=2}^{k-1} W_i)W_1$.

If we let $Q = W_1(\prod_{i=2}^{k-1} W_i)$ and $P = (\prod_{i=2}^{k-1} W_i)W_1$, left-multiply Q by Φ_1^{-1} and right-multiply Q by Φ_{k-1}^{-1},

$$\Phi_1^{-1}Q\Phi_{k-1}^{-1} = (I - \Phi_1^{-1}\frac{\nu_1 C}{C\nu_1}\Phi_1)(\prod_{i=2}^{k-2} W_i)(I - \frac{\nu_{k-1}C}{C\nu_{k-1}}) \tag{22}$$

Left-multiply P by $(I - \Phi_1^{-1}\frac{\nu_1 C}{C\nu_1}\Phi_1)$,

$$\begin{aligned} (I - \Phi_1^{-1}\frac{\nu_1 C}{C\nu_1}\Phi_1)P &= (I - \Phi_1^{-1}\frac{\nu_1 C}{C\nu_1}\Phi_1)(\prod_{i=2}^{k-1} W_i)W_1 \\ &= \Phi_1^{-1}Q\Phi_{k-1}^{-1}\Phi_{k-1}W_1 \\ &= \Phi_1^{-1}Q\Phi_1(I - \Phi_1^{-1}\frac{\nu_1 C}{C\nu_1}\Phi_1) \end{aligned} \tag{23}$$

$$\Rightarrow (I - \Phi_1^{-1}\frac{\nu_1 C}{C\nu_1}\Phi_1)P = \Phi_1^{-1}Q\Phi_1(I - \Phi_1^{-1}\frac{\nu_1 C}{C\nu_1}\Phi_1) \tag{24}$$

Let $S = (I - \Phi_1^{-1}\frac{\nu_1 C}{C\nu_1}\Phi_1)$ and $Q' = \Phi_1^{-1}Q\Phi_1(I - \Phi_1^{-1})$,

$$SP = Q'S$$

$$SPS^{-1} = Q' \tag{25}$$

As $Q' = \Phi_1^{-1}Q\Phi_1(I - \Phi_1^{-1})$ and Φ_1 is always invertible, the eigenvalues of Q' and Q have the same eigenvalues. This further implies that \$P\$ and \$Q\$ also have the same eigenvalues.

By following the same steps for perturbations at the other switching instants, we find that the eigenvalues of all the Jacobians are similar and thus the requirement for the eigenvalues of one Jacobian to be in the unit circle suffices. This completes the proof.

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TABLE I. DEFINITION OF Δ_1, Δ_2, Δ_3, AND Δ_4

$$\Delta_1 = \frac{-2z_2(0)(\lambda_1 e^{\lambda_1 L+0.5\lambda_2 T+2\lambda_1\tau} - \lambda_2 e^{\lambda_2 L+0.5\lambda_1 T+2\lambda_2\tau})}{\lambda_1\lambda_2(e^{\lambda_2 L+0.5\lambda_1 T+2\lambda_2\tau} - e^{\lambda_1 L+0.5\lambda_2 T+2\lambda_1\tau}) + 2e^{0.5(\lambda_1+\lambda_2)T}(e^{\lambda_1\tau} - e^{\lambda_2\tau})}$$

$$\Delta_2 = \frac{-2z_2(0)(\lambda_1 e^{\lambda_1 L+0.5\lambda_2 T+2\lambda_1\tau} - \lambda_2 e^{0.5\lambda_1 T+\lambda_2 L+2\lambda_2\tau})}{\lambda_1\lambda_2(e^{\lambda_2 L+\lambda_1 T+2\lambda_2\tau} - e^{\lambda_1 L+\lambda_2 T+2\lambda_1\tau} + e^{\lambda_2 L+0.5\lambda_1 T+2\lambda_2\tau} - e^{\lambda_1 L+0.5\lambda_2 T+2\lambda_1\tau}) + 2e^{0.5(\lambda_1+\lambda_2)T}(e^{\lambda_1\tau} - e^{\lambda_2\tau})}$$

$$\Delta_3 = \frac{2z_2(0)}{\lambda_1\lambda_2}$$

$$\Delta_4 = \frac{2z_2(0)(\lambda_1 e^{\lambda_1 L+0.5\lambda_2 T+2\lambda_1\tau} - \lambda_2 e^{0.5\lambda_1 T+\lambda_2 L+2\lambda_2\tau} + \lambda_2 e^{\lambda_1 T+\lambda_2 L+2\lambda_2\tau} - \lambda_1 e^{\lambda_1 L+\lambda_2 T+2\lambda_1\tau})}{\lambda_1\lambda_2(e^{\lambda_2 L+\lambda_1 T+2\lambda_2\tau} - e^{\lambda_1 L+\lambda_2 T+2\lambda_1\tau} - e^{\lambda_2 L+0.5\lambda_1 T+2\lambda_2\tau} + e^{\lambda_1 L+0.5\lambda_2 T+2\lambda_1\tau}) - 2e^{0.5(\lambda_1+\lambda_2)T}(e^{\lambda_1\tau} - e^{\lambda_2\tau})}$$

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