Topological Rankings in Communication Networks

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***Abstract* — In the theory of communication the central problem is to study how agents exchange information. This problem may be studied using the theory of connected spaces in topology, since a communication network can be modelled as a topological space such that agents can communicate if and only if they belong to the same path connected component of that space. In order to study combinatorial properties of such a communication network, notions from algebraic topology are applied. This makes it possible to determine the shape of a network by concrete invariants, e.g. the number of connected components. Elements of a network may then be ranked according to how essential their positions are in the network by considering the effect of removing them. Defining a ranking of a network which takes the individual position of each entity into account has the purpose of assigning different roles to the entities, e.g. agents, in the network. In this paper it is shown that the topology of a given network induces a ranking of the entities in the network. Furthermore, it is demonstrated how to calculate this ranking and thus how to identify weak sub-networks in any given network.**

Keywords - Ranking; Communication Networks; Topology

I. INTRODUCTION

The present paper supplements in sections VI and VII the investigation proposed in [1] with a description, in concrete examples, of the process of modelling communication networks with topological spaces.

All communication relies on the assumption that a communication link is present for exchange of information between two or more agents. The problem is that this assumption may be broken in practice, though not usually for a longer duration causing disturbances. The assumption can be interpreted as a problem of determining path connected components of a topological space. In this paper we investigate how to describe and analyze failures in communication networks, and a concrete method for doing this will be proposed. To help convey the ideas introduced here to other applications (than communication failures), a notion of communication barrier will be formulated.

The theory of connected spaces is part of the mathematical field of topology. We shall make use of notions from algebraic topology to describe communication networks in order to take advantage of algebraic structures which are apt for algorithmic description and hence more appropriate for applications. Specifically, problems will be formulated in the category of so-called abstract simplicial pairs, which are combinatorial by nature. The problem of determining the shape of such spaces is transferred to another category, namely the category of chain complexes. In this category the problem is reduced to solving a problem in linear algebra. This means that the shape of a communication network can be formulated as a rather simple problem in linear algebra by its very construction.

The theoretical framework introduced in this paper will be used to construct a ranking of the communication nodes such that the more critical nodes, measured in terms of connectedness, will receive a higher ranking. In order to define the ranking, it is necessary that the shape of the space under consideration is known. Furthermore, all elements being ranked must be comparable and hence the ranking takes place separately in each of the path components of the space.

Ranking has many applications. As an example consider assigning roles to agents with different levels of freedom according to their position in the network such that a high ranking in the network corresponds to a lower degree of freedom for the agent. If a set of agents can communicate in a meshed network topology, then each agent is assigned an intrinsic rank, which determines this agent's importance for the network to be able to communicate as a whole. For a given agent, a higher rank means that it is more likely to make communication between other agents impossible if this agent were to fail. Thus agents with a high rank can be viewed as more critical for the network than agents with a low rank. In this paper we define such a ranking system.

Section II surveys related work. In section III, the notions of categories, functors, (abstract) simplicial complexes and chain complexes are introduced. These notions will serve as the fundamental building blocks in the subsequent sections. In section IV, the theory of homology is introduced. In the following sections homology is used to determine the shapes of communication networks modelled by abstract simplicial complexes. To this end, the notion of a barrier is defined in section V, by considering the changes in the shape of a network if we remove a subset, e.g., of nodes, from the network. Also the notion of a barrier ranking will be introduced in this section. In section VI we describe a possible modelling process using the category of simplicial pairs introduced in III. Then in section VII we model parts of the gossiping problem in information theory in a concrete example and describe some weaknesses in the system via the barrier rankings introduced in V. Finally in section VIII we summarize some of our results and discuss further developments and applications.

II. EXISTING AND RELATED WORK

The topology being applied in this paper is not new. Most of the concepts are contained in [2]. A more recent textbook [3] may also be useful. A more elaborate introduction to abstract simplicial complexes can be found in [4] or [5].

Notably de Silva and Ghrist [6] describe coverage in sensor networks and proves a theorem giving a homological criterion for coverage. Various assumptions are made which are specific to sensor networks and these assumptions lead to a special way of constructing what they refer to as communication graphs. In some sense the approach taken in the present paper is similar since we too consider induced inclusion maps on homology. However, the only assumption we make for constructing a communication graph is the hypothesis: *Any pair of agents A and B can communicate*.

In [7], Ghrist and Hiraoka study similar problems of barrier conditions by sheaf cohomology. The most notable difference between the present paper and [7], besides the differences in the approach, is that Ghrist and Hiraoka study robustness by the impact on information flow whereas the present authors consider the impact on connectedness. Undoubtedly sheaf cohomology will produce more refined results, but at the expense of more complicated structures, e.g., sheaves, making the translation to the category of chain complexes a bit more obscure.

The only study the present authors were able to find, which applies methods from topology to the theory of ranking is [8], where the authors utilize a combinatorial Hodge theory, closely related to the theory of homology, to study and analyze ranking data. The methods and results in [8] seem to have a different focus from the present paper.

In the paper [9], Munch, Shapiro and Harer touch upon the subject of node failure in sensor networks. The main application is to time varying systems and they use a variant of homology, called *persistent homology*, which is designed to accommodate problems by using traditional topological methods in areas like data analysis, e.g., see [10] for a general overview.

Papers dealing with theoretical networks primarily consider graphs and make use of properties like degree of nodes etc. coming from graph theory or probability theory, see e.g., [12, 13, 14, 15]. The present paper differs from this point of view by considering higher dimensional models and other topological invariants than those in the aforementioned four publications. The term topology is used in [12, 13] to refer to network topology which in computer science is more restrictive than the notion of topology in mathematics. Here we use the term topology in the homotopy theoretical sense with homotopy equivalence as the equivalence relation, so that the star topology and any other form of tree topology is essentially the same.

III. SPACES AND CATEGORIES

We shall work in the category of abstract simplicial pairs and all possible maps between any two such pairs. The homology will be calculated via a functor to the category of chain complexes and chain maps. Both categories will be defined below and results relating their homology theories can be found in [2].

Definition III.1 (Category). A category $\mathcal{C}$ consists of a collection of objects $\mathrm{Obj}\,\mathcal{C}$, a set $\mathrm{Mor}\,\mathcal{C}$ of maps, called morphisms, between pairs of objects and for any object, an identity map, together with an associative composition of maps. A category $\mathcal{C}$ is small if $\mathrm{Obj}\,\mathcal{C}$ is a set.

One of the most studied categories is the category of topological spaces and continuous maps, abbreviated **Top**. Another important category is that of vector spaces over the field $\mathbb{F}$ with linear maps as the morphisms. The latter category is often abbreviated $\mathbf{Vec}_{\mathbb{F}}$. The abbreviation $\mathbb{F}$ will be used throughout the paper to denote a field, e.g. complex numbers, rational numbers, finite fields etc.

Roughly speaking, functors are mappings between categories that take morphisms between objects into related morphisms between related objects, and preserves identity morphisms and composition of morphisms. Functors play an essential role in topology and increasingly seem to have the potential of playing an equally big role in applied areas of mathematics, like information theory. Let $id_{\mathcal{C}}$ denote the identity map in the category $\mathcal{C}$.

Definition III.2 (Functor). Given categories $\mathcal{C}$, $\mathcal{D}$ a (covariant) functor is a map $F: \mathcal{C} \to \mathcal{D}$, such that for each object A of $\mathcal{C}$ there is assigned an object $F(A)$ of $\mathcal{D}$ and to each map f in $\mathcal{C}$ there is assigned a map $F(f)$ in $\mathcal{D}$ with the requirement that $F(id_{\mathcal{C}}) = id_{F(A)}$ and $F(f \circ g) = F(f) \circ F(g)$ whenever the composition $f \circ g$ in $\mathcal{C}$ makes sense.

An often used category in applied topology is the category with simplicial complexes as objects and simplicial maps as morphisms. Some slightly more general objects will be defined here.

Definition III.3 (Abstract simplicial complex). Let D be a discrete set. An abstract simplicial complex with 0-simplices from D is a collection X of finite subsets of D, such that for each $\sigma \in X$ all subsets of σ are also in X. A subset $\sigma \in X$ with $k+1$ elements is called a k-simplex.

A subcomplex of an abstract simplicial complex X is an abstract simplicial complex A such that every simplex in A is a simplex in X. A pair of abstract simplicial complexes is understood as a pair (X, A) where X is an abstract simplicial complex and A is a sub-complex of X. A pair of abstract simplicial complexes will be termed an abstract simplicial pair. The category of abstract simplicial pairs is the category with the abstract simplicial pairs as objects and all set maps between such pairs as morphisms. It will be abbreviated $\mathbf{Simp_a}$.

For the theory of communication, the abstract simplicial complexes often have enough structure to represent a communication network, since (k+1)-cells seldom, if ever, join at the same k-cell, in a communication network. If such a structure should be required one can model the network in a category called the (homotopy) category of CW pairs, see [2] or [3] for definitions. The only disadvantage of working with the just mentioned category is that the boundary operator in the corresponding chain complex is not generally as easy to define. Graphs in general correspond to the 1-skeleton of an abstract simplicial complex or more generally the 1-skeleton of a CW complex, see [2], [3]. The reason for not working with graphs in their usual graph theoretical setting is that higher dimensional structures can be used to model phenomena like synchronization or event ranking information. In some communication networks like those associated to electricity grids, information like Global Positioning System (GPS) timings are important features to synchronize across a grid. One idea is to define a 2-dimensional simplex with GPS timings, as a 2-simplex in which the 0-simplices have synchronized their GPS time.

Definition III.4 (Chain complexes and chain maps). A chain complex is a sequence of abelian groups, connected by morphisms such that the composition of two such morphisms is zero. A chain map is a map transforming one chain complex into another (or the same) chain complex.

The category of chain complexes, denoted by **Com** is the category with chain complexes as objects and chain maps as morphisms.

IV. HOMOLOGICAL PRELIMINARIES

In this paper the main method of computing homology $\mathrm{H}_k(X)$ of a space X is to work in the corresponding chain complex and compute its homology, see [2]. Thus the definition used here requires translating abstract simplicial complexes into chain complexes. Suppose the abstract simplicial complex X is n-dimensional.

Relative homology with coefficients in the field $\mathbb{F}$ in the category of abstract simplicial pairs is defined by the homology $\mathrm{H}_q(X, A; \mathbb{F})$ of the chain complex

$$C_q(X)/C_q(A) \xrightarrow{\partial_q} C_{q-1}(X)/C_{q-1}(A),$$

where q is a natural number (including zero) called the *index* in the chain complex. By definition $\partial_0 = 0$. Define the relative chain group as

$$C_q(X, A) = C_q(X)/C_q(A).$$

Definition IV.1 (Homology of a chain complex). The k'th homology of a pair (X, A) is given by

$$\mathrm{H}_k(X, A; \mathbb{F}) = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

Whenever $A = \emptyset$, the pair (X, A) is called absolute and the homology theory will coincide with the absolute homology theories used in applied topology. When using homology with field coefficients, the chain group $C_k(X)$ is defined as the vector space spanned by the k-simplices of X over the field $\mathbb{F}$.

The dual version of homology, called cohomology, is defined by dualizing the chain complex, i.e.

$$C^q(X)/C^q(A) \xleftarrow{\partial_q^* = \delta^{q-1}} C^{q-1}(X)/C^{q-1}(A),$$

where q is the index as before.

Definition IV.2 (Cohomology of a chain complex). The k'th cohomology is defined as

$$\mathrm{H}^k(X, A; \mathbb{F}) = \ker \delta^k / \operatorname{im} \delta^{k-1}.$$

Note that $\delta\delta = 0$ holds since $\partial\partial = 0$. Homology in the category of chain complexes (this includes cochain complexes) requires $\partial\partial = 0$ which in the category of abstract simplicial pairs can be achieved by defining the boundary operator by

$$\partial_k \sigma = \sum_{i=0}^{k} (-1)^i \sigma \,|\, [v_0, \ldots, \hat{v}_i, \ldots, v_k],$$

where $\hat{v}_i$ indicates the removal of the i'th vertex and $\sigma \,|\, [v_0, \ldots, \hat{v}_i, \ldots, v_k]$ is the restriction of σ to the corresponding face of the simplex. See [2, pp. 105-106] for a proof that $\partial\partial = 0$.

Homology and cohomology are composite functors which factors through the category of chain complexes **Com,** i.e.

$$\mathrm{H}_*, H^*: \mathbf{Simp_a} \to \mathbf{Com} \to \mathbf{Vec}_{\mathbb{F}} .$$

It should be remarked that the modelling of a communication network via abstract simplicial complexes is itself a functor.

It is important to notice that the method described here and in the next section does not change when modelling other networks. Only the functor

$$F: \mathcal{C} \to \mathbf{Simp_a} ,$$

where the category $\mathcal{C}$ can be any category, will change. For example in the case of modelling a communication network, the category $\mathcal{C}$ is defined as the category with one communication network as object and the identity map as morphism. Thus the functor F defines how to represent the communication network as an abstract simplicial complex.

In a 1-dimensional abstract simplicial complex, removing a 0-simplex corresponds to removing a row i in the matrix for ∂_1, and every column which has a non-zero element in row i. For an abstract simplicial complex X, any subset of 0-simplices is a subcomplex $A \in X$. The following version of Lefschetz duality theorem can be found in [3, p. 297].

Theorem IV.1. Given an n-dimensional abstract simplicial complex X with a subcomplex A, the following holds

1) $H_k(X,A) \cong H^{n-k}(X-A)$.
2) $H_{n-k}(X-A) \cong H^k(X,A)$.

For communication networks, the removal or death of nodes, can be analyzed via Lefschetz duality as will be shown in the next section.

V. Barriers and Rankings

In a graph X, i.e. a one dimensional abstract simplicial complex, a set of nodes A (or a set of edges) is said to be barrier significant if $\operatorname{rank} \mathrm{H}_0(X) < \operatorname{rank} \mathrm{H}_0(X-A)$. Restricting attention to the removal of 0-simplices, it is well-known how to compute $\operatorname{rank} \mathrm{H}_0(X)$; however, computing $\mathrm{H}_0(X-A)$ depends heavily on A. On the other hand computation of $\mathrm{H}^1(X,A)$ is straight forward by the associated relative chain complex described in the previous section. Since for a k-dimensional abstract simplicial complex it holds that $H_0(\mathrm{X}-\mathrm{A}) \cong \mathrm{H}^k(\mathrm{X},\mathrm{A})$ by Lefschetz duality, the choice of computing $\mathrm{H}_0(X-A)$ or $\mathrm{H}^k(X,A)$ is a matter of convenience. Some abuse of language will occur in the sense that the semantic meaning of the word barrier indicates an inaccessibility, which in the present setting only occur when a given subcomplex $A \subset X$ is removed. When a subcomplex A is barrier significant, it will simply be referred to as a barrier. Note that the inclusion map

$$i\colon (X-A) \hookrightarrow X$$

induces an injective map on homology

$$i_*: \mathrm{H}_0(X-A) \to \mathrm{H}_0(X).$$

if and only if $X-A$ and X have the same number of path components, see e.g. [2] or [3]. With this in mind we make

Definition V.1 (Barrier). The subcomplex A of the abstract simplicial complex X is called a *barrier* if the induced inclusion map $i_*: \mathrm{H}_0(X-A) \to \mathrm{H}_0(X)$ is not injective.

Let X be a 1-dimensional abstract simplicial complex and let A be a 0-dimensional subcomplex. By Lefschetz duality theorem we get that $\mathrm{H}_0(X-A) \cong \mathrm{H}^1(X,A)$. Consider the diagram

$$\begin{tikzcd} \mathrm{H}_0(X-A) \arrow[r, "i_*"] \arrow[d, "\cong"] & \mathrm{H}_0(X) \arrow[d, "\cong"] \\ \mathrm{H}^1(X,A) & \mathrm{H}^1(X). \arrow[l, "j^*"'] \end{tikzcd}$$

The induced inclusion map i_* reveals whether A constitutes a barrier or not. The dual analog j^* is induced by the inclusion map $j\colon (X,\emptyset) \hookrightarrow (X,A)$. Since we calculate homology with field coefficients, the homology groups are actually vector spaces. Hence the induced inclusion map i_* is a linear map between vector spaces. From this it is clear that A is a barrier if and only if the codimension of the kernel of i_* in $\mathrm{H}_0(X-A)$ is nonzero. In linear algebra, the map i_* may be more tricky than the map j^*, since in the latter case removing rows and columns yield smaller matrices and hence in general faster algorithms. Therefore it may be useful to consider the dual analog j^* in cohomology rather than the induced inclusion map i_* in homology.

In cohomology context, A is a barrier if and only if the codimension of the kernel of j^* in $\mathrm{H}^1(X)$ is nonzero.

Definition V.2 (Preorder). A preorder of a set S is a relation in S which is reflexive and transitive, i.e. a relation $\sim$ in a set S such that

1) $a \sim a$, for all $a \in \mathrm{S}$ (reflexivity).
2) $a \sim b \wedge b \sim c \Rightarrow a \sim c$, for $a,b,c \in \mathrm{S}$ (transitivity).

A preorder on S is total if for every $a,b \in \mathrm{S}$ it holds that $a \sim b$ or $b \sim a$. It is now possible to define what is meant by a ranking of a set.

Definition V.3 (Ranking). A ranking of a set S is a total preorder $\sim$ on the set S.

Let X be a 1-dimensional abstract simplicial complex. A ranking of the 0-skeleton X^0 of X, called a 0-ranking, is an assignment of numbers to the 0-simplices according to their barrier significance together with the relation $\alpha \sim \beta$ iff. $a_\alpha \le b_\beta$ where a_α, b_β are the ranks associated to the 0-simplices α and β respectively. Precisely, to each 0-simplex γ of X^0, we associate the *rank* a_γ given as the difference

$$a_\gamma = \dim \mathrm{H}_0(X-\{\gamma\}) - \dim \mathrm{H}_0(X).$$

More generally if X is an n-dimensional abstract simplicial complex, then a ranking of the simplices in the s-skeleton X^s of X can be defined, called an s-ranking.

If the cardinality of the finite set of simplices we remove is greater than one, say $k > 1$, then the s-ranking is said to be of order k. In other words, an s-ranking of order k is a ranking of the s-skeleton X^s where the rank associated with each k-tuple $\{\gamma_0,\ldots,\gamma_{k-1}\}$ of simplices in X^s, is given by the difference

$$\dim \mathrm{H}_0(X - \{\gamma_0,\ldots,\gamma_{k-1}\}) - \dim \mathrm{H}_0(X).$$

We get

Proposition 1. Given an n-dimensional abstract simplicial complex X. Then there exists a (barrier) s-ranking of order k of X, for all k less than the number of simplices in X^s.

The order of an s-ranking thus defines the number of s-simplices that are removed. The ranking described here will be called a *barrier ranking*.

VI. Modelling Networks with Simplicial Pairs

Consider a category $\mathcal{C}$ dwith one object and only the identity map. Suppose the object is exactly what is to be modelled, e.g., an electrical grid, a telecommunication network etc. Depending on the purpose of the system, it may be very complicated and often impossible to describe this in every detail. The modelling process has as its main purpose to extract the details necessary for the given application, e.g., potentials and currents in an electrical grid. The main ingredient in the modelling process is to construct a functor from $\mathcal{C}$ to some other category $\mathcal{D}$ such that desired properties are preserved and made more tangible.

Loosely speaking, the modelling process consists of three steps. Define an initial category $\mathcal{C}$ for the system to be modelled. Secondly determine what kind of features should be taken into account and define the target category $\mathcal{D}$ in accordance with this. The last step is to define a faithful functor $\mathcal{C} \to \mathcal{D}$ translating objects and maps in $\mathcal{C}$ to objects and maps in $\mathcal{D}$.

The kind of homology functor which we have described in this paper so far, is one that factors through a category where the objects are simplicial pairs and the set of morphisms is all maps between such pairs. We have shown how to extract useful information via the homology functor, but we have not yet described the modelling process which is done prior to using the homology functor.

In the following we shall describe a functor $F: \mathcal{D} \to \mathbf{Simp_a}$ from a model category $\mathcal{D}$ to the category of simplicial pairs $\mathbf{Simp_a}$. Using the generic functor F, composed with H_*, we can extract useful information from $\mathcal{D}$.

In case of modelling power systems like transmission grids or distribution grids with $\mathbf{Simp_a}$, the usual information of interest is suitably modelled by 1-dimensional simplicial pairs. Transmission lines could be modelled as 1-dimensional simplices and transformer stations as 0-dimensional simplices, i.e., the grid is modelled as a graph. In this case the ranking described earlier is a purely mathematical property of the grid, where high ranks indicate critical simplices in the model. Obviously, it is necessary that there exists a faithful transfer of critical information from $\mathcal{C}$, via $\mathcal{D}$, to $\mathbf{Simp_a}$. This can be ensured by substituting the category $\mathcal{D}$ with a more suitable one; if such a category exists.

An example of such a choice of $\mathcal{D}$, follow an approach credited to Maxwell for finding currents and voltages in an electrical circuit. The approach is often called the Mesh Current Method and is roughly an algorithmic way to a homology description of $\mathcal{D}$, where $\mathcal{D}$ is chosen to be the category with designs of electrical circuits as objects and all changes in circuit designs as morphisms. With this $\mathcal{D}$, there is a very obvious connection to $\mathbf{Simp_a}$, in which the diagrammatic approach in circuit theory is easily translated into a weighted graph, i.e. into a 1-dimensional absolute simplicial pair $(X, \emptyset)$ and from there to the chain groups $C(X)$ (with a suitable choice of coefficient field) associated with X. This is well-known and the reader is referred to [11] for a comprehensive description of the connection between electrical circuits and homology.

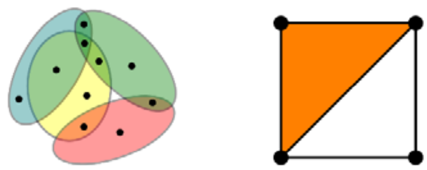
In order to facilitate intuition of modelling with 2-dimensional simplicial complexes and more generally in higher dimensions, we introduce the following

Definition VI.1 (Nerve of a family of sets). Let $S = (S_i)_{i\in I}$ be a family of sets where I is an arbitary index set. Then the *nerve* of $S = (S_i)_{i\in I}$ is the simplicial complex $\mathcal{N}(S)$ whose simplices are finite collections of non-empty sets from S with non-empty intersections. Thus the vertices of $\mathcal{N}(S)$ are the non-empty sets from $S = (S_i)_{i\in I}$.

Notice that the nerve of a family of sets is a functor

$$\mathcal{N}: \mathbf{FinSet} \to \mathbf{Simp_a}$$

from the category of finite sets **FinSet** to the category $\mathbf{Simp_a}$ of simplicial pairs. In figure 1, a family of sets S is shown together with the nerve $\mathcal{N}(S)$ of S.



(a) S (b) $\mathcal{N}(S)$

Figure 1: Nerve of a family of sets

There is one vertex in $\mathcal{N}(S)$ for each set in S. The edges in $\mathcal{N}(S)$ corresponds to pairs (S_i, S_j) in S such that $S_i \cap S_j \neq \emptyset$. Finally the face in $\mathcal{N}(S)$ correspond to the triple (S_i, S_j, S_k) in S such that $S_i \cap S_j \cap S_k \neq \emptyset$. When modelling higher dimensional phenomena where there is an underlying topological space X and the family $S = (S_i)_{i\in I}$ is a family of subsets of X, then it is often easier to consider relations by finite intersections of the subsets rather than directly modelling with simplicial complexes (or even pairs). The corresponding homology theories, however, requires more machinery, which often can be avoided, by applying a relatively simple but powerful lemma, called the *Nerve lemma*, see e.g., [3] for a proof.

Lemma VI.1 (Nerve lemma). Let $\mathcal{N}(S)$ be the nerve of some family of subsets $S = (S_i)_{i \in I}$ in a topological space X where all non-empty intersections of subsets from S are contractible. Then $\mathcal{N}(S)$ is homotopy equivalent to X.

VII. Ranking in the Gossiping Problem

The gossiping problem in information theory is to decide whether a group of agents, each knowing a unique piece of information, can communicate this information to everyone else according to a given set of rules, e.g., delivering all accumulated information via telephone. It is often of interest to find the minimum number of communication instances needed to solve the problem, e.g., the number of telephone calls etc. In this section we study in a concrete example the connection between the notion of gossiping in information theory and the barrier ranking as introduced in this paper.

The gossiping problem, together with many other problems in applied mathematics, assume that the communication graph is connected. The barrier ranking reveals whether this is the case when an agent stops communicating. Moreover, barrier ranking on the nerve of a covering can reveal the critical intersections (sets of agents) which are necessary for a connected communication system.

With reference to figure 2 we consider the gossiping problem for a network with nine agents. The communication protocol is described via a family $\mathcal{F}$ of four sets S_1, S_2, S_3, S_4 and the rule that agents can communicate if and only if they belong to the same set.

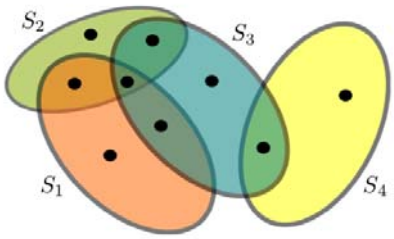


Figure 2: $\mathcal{F}$ and the nine agents

The nerve of this family of sets is then given by

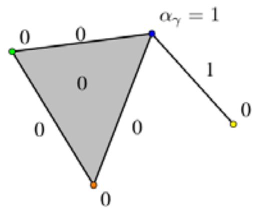


Figure 3: The nerve of $\mathcal{F}$ and the rank of all elements

Following the terminology introduced here, whenever two agents belonging to the same set in $\mathcal{F}$ stop being able to communicate, then the result is that one of them is excluded from the set. When the possibility of communication between two agents from distinct sets in $\mathcal{F}$ cease to exist it means that the finite intersection of the sets is empty.

For the nerve of the family $\mathcal{F}$ shown in figure 3, the 0, 1 and 2 ranking is marked by numbers at the simplices. These numbers are easy to find using the figure, but how to calculate them automatically is another matter.

Let $\mathcal{N}(\mathcal{F})$ denote the simplicial complex in figure 3. Then the chain complex for $\mathcal{N}(S)$ with coefficient field $\mathbb{F} = \mathbb{Z}_2$ the cyclic group of order 2, is given by

$$0 \xrightarrow{\partial_3 = 0} \mathbb{F} \xrightarrow{\partial_2} \mathbb{F}^4 \xrightarrow{\partial_1} \mathbb{F}^4 \xrightarrow{\partial_0 = 0} 0,$$

where ∂_2 and ∂_1 are non-trivial maps. Since homology in dimension zero only depends on the 1-skeleton of $\mathcal{N}(\mathcal{F})$, the linear map ∂_2 is not relevant with respect to the barrier ranking.

The boundary operator ∂_1 is a linear map between vector spaces over $\mathbb{F}$. After row operations the boundary operator ∂_1 is equivalent to

$$\partial_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The homology group $\mathrm{H}_0(\mathcal{N}(\mathcal{F}))$ can then be calculated as

$$\mathrm{H}_0(\mathcal{N}(\mathcal{F})) = \ker \partial_0 / \operatorname{im} \partial_1 = \mathbb{F}^4/\mathbb{F}^3 \cong \mathbb{F},$$

which reflects that the communication graph is connected.

There is only one non-trivial 0-ranking which occurs when we remove the vertex γ in the nerve $\mathcal{N}(\mathcal{F})$. The rank α_γ of γ is computed by considering the chain complex for the nerve $\mathcal{N}_\gamma$ of the family $\mathcal{F}$ with the set S_3 corresponding to γ removed. We get the chain complex

$$0 \xrightarrow{\partial_2 = 0} \mathbb{F} \xrightarrow{\partial_1} \mathbb{F}^3 \xrightarrow{\partial_0 = 0} 0,$$

in which the boundary operator ∂_1 is given by

$$\partial_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

From this we find

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$$\mathrm{H}_0(\mathcal{N}_\gamma) = \mathbb{F}\oplus\mathbb{F},$$

giving the rank of γ

$$\alpha_\gamma = \operatorname{rank}\mathrm{H}_0(\mathcal{N}_\gamma) - \operatorname{rank}\mathrm{H}_0(\mathcal{N}(\mathcal{F})) = 1.$$

VIII. Discussion and Conclusion

In this paper we have described a modelling process involving topological spaces and combinatorial objects like simplicial pairs, and how they efficiently describe various levels in a communication network. Most of the features of communication networks which can suitably be described by topological spaces are intrinsic features.

It was shown how to construct a ranking system based on information about the shape of a communication system. A natural next step would be to consider real examples and determine if irregularities, e.g. from noise etc., will make such an analysis impossible. In this case it may be necessary to consider the notion of persistent (co)homology [10] and investigate if the notions of barriers and rankings used in such a setting can handle extensions of this character. Noise might give vastly different information about the irregularities depending on the dimension of the model, much like barrier ranking of higher dimensional models may give new and different information about the intrinsic features of a communication network.

The authors believe that there exists a similar kind of barrier ranking when using homology groups in higher dimensions and that these should be studied further. Such rankings might give important information about the topology of the given model.

All our investigations of simplicial (co)homology theory so far have been made with coefficients in a field. It is possible that homology theories with non-field coefficients will shed new light on already investigated topics.

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