Non-Equidistant Multi-Variable Optimum Model with Fractional Order Accumulation Based on Vector Continued Fractions Theory and its Application

Qiyuan LIU[1, 2, *], Dejie YU[1]

1. State key Laboratory of Advanced Design and Manufacture for Vehicle Body, Hunan University, Changsha, Hunan, 410082, China
2. College of Mechanical Engineering, Hunan University of Arts and Science, Changde, 415000, China

***Abstract* — Grey system is a theory which specially studies poor information, and it possesses wide suitability. The parameters of multivariable grey FMGM(1,n) model with the fractional order accumulation can reflect the restricted and affected multi-variables relation with one another. We start by constructing background values based on vector continued fractions theory by using rational interpolation with trapezoidal rule in numerical integration and extrapolation formula, then establish a non-equidistant multi-variable grey model NFMGM(1,n) with the fractional order accumulation. Taking the mean relative error as objective function and taking the modified values of response function initial values and the number of fractional order as design variables, we build a multi-variable non-equidistance grey optimizing model ONFMGM(1,n) with a fractional order accumulation. The model is high precision and easy to use, it is not only suitable for equidistant modelling but also for non-equidistant modelling. Examples validate the practicability and reliability of the proposed model.**

Keywords - multivariable; fractional order accumulation operator; non-equidistance sequence; vector continued fractions theory; ONFMGM(1,n)

I. INTRODUCTION

Grey system is a theory which studies poor information specially and it possesses wide suitability. Grey model as an important part of the grey system theory has been widely used in many fields since the grey system theory was put forward [1-3]. The actual systems are mostly fractional order. That using fractional order to describe the object with fractional order property can better reveal their nature and behavior. The grey model FGM (1,1) with fractional order accumulation was proposed [4], and the weapons maintenance fee was predicted using FGM (1,1) [5]. The discrete grey model DFGM (1,1) with fractional order accumulation was proposed [6]. FGM (1,1) and DFGM (1,1) was summarized and the applicable range of FGM (1,1) was researched [7]. The optimal solution of FGM (1,1) was studied [8]. The non-homogenous grey model NFGM(1,n) with the fractional order accumulation was put forward [9], and this model widens the grey model with the fractional order. It is more difficult to solve the grey model with fractional order than with integer order. Actual systems often contain multiple variables, and they are most non-equidistance. It has important theoretical and practical value how to establish and solve non-equidistant multivariate grey model with the fractional order accumulation. Construction method about background value is a key factor affecting prediction accuracy and adaptability. In order to improve the fitting and prediction accuracy of GM(1,1), a variety of construction methods about background value were put forward and some non-equidistant GM(1,1) models were constructed in [10-13]. In this paper, constructing background value based on vector continued fractions theory by using rational interpolation, trapezoidal rule in numerical integration and extrapolation formula, the non-equidistant multi-variable grey model NFMGM(1,n) with the fractional order accumulation was established. Taking the mean relative error as objective function and taking the modified values of response function initial values and the number of fractional order as design variables, multi-variable non-equidistance grey optimizing model ONFMGM(1,n) with the fractional order accumulation was built. The model with high precision and easy to use is not only suitable for equidistant modeling, but also for non-equidistant modeling. Examples validate the proposed model with the practicability and reliability has important theoretical and practical value.

II. NON-EQUIDISTANT MULTI-VARIABLE OPTIMUM MODEL WITH FRACTIONAL ORDER ACCUMULATION BASED ON VECTOR CONTINUED FRACTIONS THEORY

Definition 1: Supposed the sequence $X_i^{(0)} = [x_i^{(0)}(t_1), \cdots, x_i^{(0)}(t_j), \cdots, x_i^{(0)}(t_m)]$, if $\Delta t_j = t_j - t_{j-1} \neq const$ where $i = 1,2,\cdots,n$, $j = 2,\cdots,m$, n is the number of variables and m is the sequence number of each variable, $\mathbf{X}_i^{(0)}$ is called as non-equidistant sequence.

Definition 2: Supposed the sequence $\mathbf{X}_i^{(1)} = \{x_i^{(1)}(t_1), x_i^{(1)}(t_2), \cdots, x_i^{(1)}(t_j), \cdots, x_{t_j}^{(1)}(t_m)\}$ if

$x_i^{(1)}(t_1) = x_i^{(0)}(t_1)$ and $x_i^{(1)}(t_j) = x_i^{(1)}(t_{j-1}) + x_i^{(0)}(t_j) \cdot \Delta t_j$ where $j = 2,\cdots,m,\ i = 1,2,\cdots,n,$ and $\Delta t_j = t_j - t_{j-1}$, $\mathbf{X}_i^{(1)}$ is one-time accumulated generation operation of non-equidistant sequence $\mathbf{X}_i^{(0)}$, and it is denoted by 1-AGO.

Definition 3 [9]: Supposed the non-negative sequence $X_i^{(0)} = [x_i^{(0)}(t_1),\cdots,x_i^{(0)}(t_j),\cdots,x_i^{(0)}(t_m)]$, if $\Delta t_i = t_i - t_{i-1} \neq const$, $X_i^{(r)} = [x_i^{(r)}(t_1)\ \ x_i^{(r)}(t_2)\ \ \cdots\ \ x_i^{(r)}(t_m)]$ is called as r-order accumulated generation operation of $X_i^{(0)}$, where

$$x_i^{(r)}(t_k) = \sum_{i_1=1}^{k} x_i^{(r-1)}(t_{i_1})\Delta t_{i_1} = \begin{cases} x_i^{(r)}(t_{k-1}) + x_i^{(r-1)}(t_k)\Delta t_k, k = 2,3,\cdots,m \\ x_i^{(r)}(t_1) = x_i^{(0)}(t_1) \qquad k = 1 \end{cases} \tag{1}$$

The following equation can be deduced by using the above definitions and the inverse accumulated generation operation [9].

$$x_i^{(r)}(t_k) = \sum_{i_1=1}^{k} \frac{\Gamma(r+k-1)}{\Gamma(k-i_1+1)\Gamma(r)} (x^{(0)}(t_{i_1})\Delta t_{i_1} \tag{2}$$

where, $k = 1,2,\cdots,m$, and Γ is Gamma function.

Note: Eq.(1) is established when r is an integer, and Eq.(2) is obtained when r is extended from integer to fraction, so Eq.(2) should be used when calculating the fractional $x_i^{(r)}(t_k)$.

Supposed the multivariable original data matrix:

$$X^{(0)} = \{X_1^{(0)}, X_2^{(0)},\cdots,X_n^{(0)}\}^{\mathbf{T}} = \begin{bmatrix} x_1^{(0)}(t_1) & x_1^{(0)}(t_2) & \cdots & x_1^{(0)}(t_m) \\ x_2^{(0)}(t_1) & x_2^{(0)}(t_2) & \cdots & x_2^{(0)}(t_m) \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{(0)}(t_1) & x_n^{(0)}(t_2) & \cdots & x_n^{(0)}(t_m) \end{bmatrix} \tag{3}$$

where $\mathbf{X}^{(0)}(t_j) = [x_1^{(0)}(t_j), x_2^{(0)}(t_j),\cdots,x_n^{(0)}(t_j)]$ is the observation value of each variable at t_j, and the sequence $[x_i^{(0)}(t_1), x_i^{(0)}(t_2),\cdots,x_i^{(0)}(t_j),\cdots,x_i^{(0)}(t_m)]$ is non-equidistant, that is, the distance $t_j - t_{j-1}$ is not constant.

In order to establish the model, firstly the original data is accumulated r times to generate a new matrix as:

$$X^{(r)} = \{X_1^{(r)}, X_2^{(r)},\cdots,X_n^{(r)}\}^{\mathbf{T}} = \begin{bmatrix} x_1^{(r)}(t_1) & x_1^{(r)}(t_2) & \cdots & x_1^{(r)}(t_m) \\ x_2^{(r)}(t_1) & x_2^{(r)}(t_2) & \cdots & x_2^{(r)}(t_m) \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{(r)}(t_1) & x_n^{(r)}(t_2) & \cdots & x_n^{(r)}(t_m) \end{bmatrix} \tag{4}$$

where, $x_i^{(r)}(t_j)$ meets the conditions in the definition 3.

Non-equidistant multi-variable model NFMGM(1,n) with the fractional order accumulation can be expressed as r-order differential equations with n variables:

$$\begin{cases} \frac{dx_1^{(r)}}{dt} = a_{11}z_1^{(r)} + a_{12}z_2^{(r)} + \cdots + a_{1n}z_n^{(r)} + b_1 \\ \frac{dx_2^{(r)}}{dt} = a_{21}z_1^{(r)} + a_{22}z_2^{(r)} + \cdots + a_{2n}z_n^{(r)} + b_2 \\ \cdots \\ \frac{dx_n^{(r)}}{dt} = a_{n1}z_1^{(r)} + a_{n2}z_2^{(r)} + \cdots + a_{nn}z_n^{(r)} + b_n \end{cases}$$

The albino differential equations can be expressed as:

$$\begin{cases} \frac{dx_1^{(r)}}{dt} = a_{11}x_1^{(r)} + a_{12}x_2^{(r)} + \cdots + a_{1n}x_n^{(r)} + b_1 \\ \frac{dx_2^{(r)}}{dt} = a_{21}x_1^{(r)} + a_{22}x_2^{(r)} + \cdots + a_{2n}x_n^{(r)} + b_2 \\ \cdots \\ \frac{dx_n^{(r)}}{dt} = a_{n1}x_1^{(r)} + a_{n2}x_2^{(r)} + \cdots + a_{nn}x_n^{(r)} + b_n \end{cases} \tag{5}$$

Assumed $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$, Eq.(5) can be expressed as:

$$\frac{dX^{(r)}(t)}{dt} = AX^{(r)}(t) + B \tag{6}$$

Regarded the first component of $x_i^{(r)}(t_j)$ as the initial conditions of the grey differential equation, the continuous time response of Eq.(6) is as:

$$X^{(r)}(t) = e^{At}X^{(r)}(t_1) + A^{-1}(e^{At} - I)B \tag{7}$$

In Eq.(7), the first column of data is taken as the initial value of the solution, and $X_i^{(0)}(t_1) + \beta_i$ is taken the place of $X_i^{(0)}(t_1)$, where β is a vector whose dimension is equal to $X^{(0)}(t_1)$, that is, $\hat{X}^{(r)}$. $\hat{X}^{(r)}$ is restored as the fitting value of the original sequence $\hat{X}^{(0)}$ to derive the calculation formula of $\hat{X}^{(0)}$ [9].

$$\hat{x}_i^{(0)}(t_k) = \sum_{i_1=0}^{k-1} (-1)^i \frac{\Gamma(r+1)}{\Gamma(i_1+1)\Gamma(r-i_1+1)} \hat{x}_i^{(r)}(t_{k-i}) / \Delta t_k \tag{8}$$

In order to identify $\mathbf{A}$ and B, Eq.(5) is discretized and the following equation can be obtained:

$$x_i^{(r-1)}(t_j)\Delta t_j = \sum_{l=1}^{n} a_{il} \int_{t_{j-1}}^{t_j} x_i^{(r)}(t_j)dt + b_i \Delta t_j$$

$$x_i^{(r-1)}(t_j) = \sum_{l=1}^{n} a_{il} \frac{\int_{t_{j-1}}^{t_j} x_i^{(r)}(t_j)dt}{\Delta t_j} + b_i \tag{9}$$

where $i = 1,2,\cdots,n$ and $j = 2,3,\cdots,m$.

Assumed $a_i = (a_{i1}, a_{i2}, \cdots, a_{in}, b_i)^{\mathrm{T}}$, the identified value $\hat{\mathbf{a}}_i$ of a_i can be obtained through the least square method:

$$\hat{a}_i = [\hat{a}_{i1}, \hat{a}_{i2}, \cdots, \hat{a}_{in}, \hat{b}_i]^{\mathrm{T}} = (L^T L)^{-1} L^T Y_i \tag{10}$$

Supposed $z_i^{(1)}(t_j) = \frac{\int_{t_{j-1}}^{t_j} x_i^{(1)}(t_j)dt}{\Delta t_j}$, when the background value is generated by mean value, that is, $z_i^{(r)}(t_j) = 0.5(x_i^{(r)}(t_{j-1}) + x_i^{(r)}(t_{j-1}))$, the expression of $\mathbf{L}$ is as follows:

$$L = \begin{bmatrix} \frac{1}{2}(x_1^{(r)}(t_1)+x_1^{(r)}(t_2)) & \frac{1}{2}(x_2^{(r)}(t_1)+x_{2_1}^{(r)}(t_3)) & \cdots & \frac{1}{2}(x_n^{(r)}(t_1)+x_{n_1}^{(r)}(t_2)) & 1 \\ \frac{1}{2}(x_1^{(r)}(t_2)+x_1^{(r)}(t_3)) & \frac{1}{2}(x_2^{(r)}(t_2)+x_2^{(r)}(t_3)) & \cdots & \frac{1}{2}(x_n^{(r)}(t_2)+x_n^{(r)}(t_3) & 1 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ \frac{1}{2}(x_1^{(r)}(t_{m-1})+x_1^{(r)}(t_m)) & \frac{1}{2}(x_2^{(r)}(t_{m-1})+x_2^{(r)}(t_m)) & \cdots & \frac{1}{2}(x_n^{(r)}(t_{m-1})+x_n^{(r)}(t_m)) & 1 \end{bmatrix} \tag{11}$$

$$Y_i = [x_i^{(r-1)}(t_2), x_i^{(r-1)}(t_3), \cdots, x_i^{(r-1)}(t_m)]^{\mathrm{T}} \tag{12}$$

The background value in non-equidistant multi-variable model NFMGM(1,n) with the fractional order accumulation is generated by mean value so as to bring about lower accuracy, so a method of reconstructing background value was put forward which was based on vector continued fractions theory by using rational interpolation, trapezoidal rule in numerical integration and extrapolation formula so as to improve the accuracy of the model [14]. The constructing method of background value is as follows:

Definition 3 [9]: Assumed that $\{a_n\}$ and $\{b_n\}$ are two real series, the fraction whose form as $b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}}$ is called as the continued fraction that noted as $b_0 + \underset{n=1}{\overset{\infty}{K}}(a_n / b_n)$ and the formula as $b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots + \cfrac{a_n}{b_n}}}}}$ is called as n-order progressive fractional of the continued fraction.

Definition 4 [14]: Assumed that $v = (v_1, v_2, \cdots, v_n)$ is a d-dimensional vector and $|v| = \sqrt{\sum_{j=1}^{d} v_j^2}$ is the modulus of the vector, the generalized inverse is $v^{-1} = \frac{v}{|v|^2}$.

Definition 5 [9]: Assumed $\varphi[x_i] = v_i, i = 0,1,\cdots$, $\varphi[x_p, x_q] = \frac{x_q - x_p}{\varphi[x_q] - \varphi[x_p]}$ and $\varphi[x_i, \cdots x_j, x_k, x_l] = \frac{x_l - x_k}{\varphi[x_i, \cdots, x_j, x_l] - \varphi[x_i, \cdots, x_j, x_k]}$, $\varphi[x_i, \cdots x_j, x_k, x_l]$ is the inverse difference quotient of the l-order vector of vector set V^m in $x_0, x_1, \cdots, x_l$.

Theorem 1: Setting

$$R_n(x) = \varphi[x_0] + \cfrac{x - x_0}{\varphi[x_0, x_1] + \cfrac{x - x_1}{\varphi[x_0, x_1, x_2] + \ddots + \cfrac{x - x_{n-1}}{\varphi[x_0, x_1, x_2, \cdots, x_n]}}},$$

where $\varphi[x_0, x_1, \cdots, x_k] \neq 0, \infty, k = 0,1,\cdots,n$ is the inverse difference quotient of the k-order vector of vector set V^m in $x_0, x_1, \cdots, x_k$, and $\varphi[x_i, \cdots x_j, x_k, x_l]$ is the inverse difference quotient of the l-order vector of vector set V^m in $x_0, x_1, \cdots, x_l$, we can obtain:

$$R_n(x_i) = v_i = (x_1^{(r)}(i), x_2^{(r)}(i), \cdots, x_n^{(r)}(i)), i = 0,1,\cdots,n$$

Setting the integration interval $[a,b]$ is divided into m_1 equal portions, the step length is $h = (b-a)/m_1$, and then the generalized trapezoidal integration formula is as:

$$T_m(f) = h[\frac{1}{2}f(a) + f(a+h) + \cdots + f(a+(m_1-1)h + \frac{1}{2}f(b)]$$

When $m_1 = 4$, $zf^{(r)}(k+1) = \frac{1}{4}[\frac{1}{2}x^{(r)}(k) + \cdots + x^{(r)}(k+\frac{3}{4}) + \frac{1}{2}x^{(r)}(k+1)]$

When $m_1 = 8$, $ze^{(r)}(k+1) = \frac{1}{8}[\frac{1}{2}x^{(r)}(k) + \cdots + \frac{1}{2}x^{(r)}(k+1)]$

The combination formula is:

$$z^{(r)}(k+1) = \frac{4}{3}ze^{(r)}(k+1) - \frac{1}{3}zf^{(r)}(k+1) \quad k = 1,2,\cdots,m-1$$

$z_i^{(r)}(t_{j+1}) = \frac{4}{3}ze_i^{(r)}(t_{j+1}) - \frac{1}{3}zf_i^{(r)}(t_{j+1})$ where $j = 1,2,\cdots,m-1$ as the background value of the grey derivative vector and noting $a_i = (a_{i1}, a_{i2}, \cdots, a_{in}, b_i)^{\mathbf{T}}$, the identified value $\hat{a}_i$ of a_i can be obtained by using the least square method:

$$\hat{a}_i = (\hat{a}_{i1}, \hat{a}_{i2}, \cdots, \hat{a}_{in}, \hat{b}_i)^{\mathbf{T}} = (L^{\mathbf{T}}L)^{-1}L^{\mathbf{T}}Y_i$$

where,

$$L = \begin{bmatrix} \frac{4}{3}ze_1^{(r)}(t_2) - \frac{1}{3}zf_1^{(r)}(t_2) & \frac{4}{3}ze_2^{(r)}(t_2) - \frac{1}{3}zf_2^{(r)}(t_2) & \cdots & \frac{4}{3}ze_n^{(r)}(t_2) - \frac{1}{3}zf_n^{(r)}(t_2) & 1 \\ \frac{4}{3}ze_1^{(r)}(t_3) - \frac{1}{3}zf_1^{(r)}(t_3) & \frac{4}{3}ze_2^{(r)}(t_3) - \frac{1}{3}zf_2^{(r)}(t_3) & \cdots & \frac{4}{3}ze_n^{(r)}(t_3) - \frac{1}{3}zf_n^{(r)}(t_3) & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{4}{3}ze_1^{(r)}(t_m) - \frac{1}{3}zf_1^{(r)}(t_m) & \frac{4}{3}ze_2^{(r)}(t_m) - \frac{1}{3}zf_2^{(r)}(t_m) & \cdots & \frac{4}{3}ze_n^{(r)}(t_m) - \frac{1}{3}zf_n^{(r)}(t_m) & 1 \end{bmatrix} \quad (13)$$

Then the identified values of A and B can be get:

$$\hat{A} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{a}_{n1} & \hat{a}_{n2} & \cdots & \hat{a}_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \cdots \\ \hat{b}_n \end{bmatrix} \quad (14)$$

The calculated value in NFMGM(1,n) is:

$$\hat{X}_i^{(r)}(t_j) = e^{\hat{A}(t_j - t_1)}X^{(1)}(t_1) + \hat{A}^{-1}(e^{\hat{A}(t_j - t_1)} - I)\hat{B} \quad (15)$$

After restoring the fitting value of the original data can be get:

$$\hat{x}_i^{(0)}(t_k) = \sum_{i_1=0}^{k-1}(-1)^i \frac{\Gamma(r+1)}{\Gamma(i_1+1)\Gamma(r-i_1+1)}\hat{x}_i^{(r)}(t_{k-i})/\Delta t_k \quad (16)$$

The absolute error of the ith variable:

$$\hat{x}_i^{(0)}(t_j) - x_i^{(0)}(t_j)$$

The relative error of the ith variable (%):

$$e_i(t_j) = \frac{\hat{x}_i^{(0)}(t_j) - x_i^{(0)}(t_j)}{x_i^{(0)}(t_j)} * 100$$

The mean of the relative error of the ith variable:

$$\frac{1}{m}\sum_{j=1}^{m}\left|e_i(t_j)\right|$$

The average error of the whole data:

$$f = \frac{1}{nm}\sum_{i=1}^{n}(\sum_{j=1}^{m}\left|e_i(t_j)\right|) \tag{17}$$

Taking the average error f as the objective function, and β and the fractional order r as design variables, it is obtained by using genetic optimization function ga in Matlab2015b or other optimization methods, and then the model is tested. The optimization model of NFMGM (1, n) is taken as ONFMGM (1, n).

It can be seen that NFMGM(1,n) is degraded to the non-equidistant NFGM(1,n) model when $n=1$, and NFMGM(1,n) is degraded to the non-equidistant multi-variable NMGM(1,n) model when r=1. NFMGM(1,n) can be used to build model in non-equidistant and equidistant time sequences. In the case of different values of n, the different non-equidistant models can be obtained, such as NFMGM (1,2), NFMGM (1,3) and NFMGM (1,4).

III. APPLICATION EXAMPLES OF MODEL

Example 1: YT14 cemented carbide tool was used to turning cylindrical in CA6140 lathe. When the tool geometry and cutting speed were constant, cutting depth was changed. Experiment data about the measured cutting force are as shown in Table 1 [15]:

TABLE 1 EXPERIMENT DATA IN CUTTING (F=0.02MM/R)

No.	1	2	3	4	5
a_p / mm	1.00	1.25	1.50	1.75	2.00
F_{1z} / N	838.98	1060.45	1261.79	1483.25	1704.72
F_{1y} / N	255.10	290.16	355.22	420.28	469.08

Assumed cutting depth a_p as t_k , main cutting force F_{1z} as x_1 and F_{1y} as x_2 , the multi-variable non-equidistance grey optimizing model ONFMGM(1,2) with the fractional order accumulation was established by using the proposed method in this paper. The parameters of this model are as follows:

$A=\begin{vmatrix} -8.1596 & 31.0935 \\ -2.6935 & 10.1067 \end{vmatrix}$, $B=10^3\begin{vmatrix} 2.7602 \\ 0.8277 \end{vmatrix}$, $\beta=\begin{vmatrix} 0.43687 \\ 0.45805 \end{vmatrix}$, r=2.0025

The fitting value of F_{1y} :

$\hat{F}_{1y}$ = [255.55805,288.02018,363.51909,420.2386,469.07998]

The absolute error of F_{1y} :

q =[0.45805,-2.1398,8.2991,-0.0414,-1.9608e-05]

The relative error (%) of F_{1y} :

e = [0.17956,-0.73746,2.3363,-0.0098506,-4.1802e-06]

The average relative error of F_{1z} is 0.53123%, and one of F_{1y} is 0.65264%. The average relative error of this model is 0.59194%. Thus, the accuracy of the model is higher.

When r = 1 and not optimizing β , the average relative error of F_{1z} is 0.59422%, one of F_{1y} is 1.5733%, and the average relative error of this model is 1.0837%.

Example 2: The affecting data of water absorption to mechanical properties of pure PA66 are seen in [16]. After PA66 samples with the different water absorption were tested in mechanical property, the following experimental data of PA66 can be obtained, as shown in Table 2 and Table 3, where $X_1^{(0)}$ is bending strength (Mpa), $X_2^{(0)}$ is bending elastic modulus (Gpa) and $X_3^{(0)}$ is tensile strength (Mpa).

TABLE 2 THE AFFECTING DATA OF WATER ABSORPTION TO MECHANICAL PROPERTIES OF PA66 SAMPLES FROM NO.1 TO NO.5

No.	1	2	3	4	5
water absorption t_j / %	0	0.0607	0.1071	0.1662	0.2069
$X_1^{(0)}$	83.4	84.9	84.5	84.2	84.4
$X_2^{(0)}$	2.63	2.64	2.61	2.65	2.66
$X_3^{(0)}$	84.2	84.4	86.3	84.3	81.3

TABLE3 THE AFFECTING DATA OF WATER ABSORPTION TO MECHANICAL PROPERTIES OF PA66 SAMPLES FROM NO.6 TO NO.9

No.	6	7	8	9
water absorption t_j / %	0.4344	0.5243	0.8524	0.9756
$X_1^{(0)}$	78.4	75.4	59.5	54.1
$X_2^{(0)}$	2.52	2.32	1.90	1.72
$X_3^{(0)}$	74.9	75.7	73.2	66.9

The multi-variable non-equidistance grey optimizing model ONFMGM(1,3) with the fractional order accumulation was established by using the proposed method in this paper. The parameters of this model are as follows:

$$A = \begin{vmatrix} 0.4530 & -1.1729 & -0.9483 \\ 0.0081 & -0.0161 & -0.0244 \\ -0.1517 & -0.6811 & -0.2182 \end{vmatrix}, \quad B = \begin{vmatrix} 140.6140 \\ 4.4164 \\ 127.4190 \end{vmatrix},$$

$$\beta = \begin{vmatrix} -6.1619 \times 10^{-8} \\ 5.9829 \times 10^{-6} \\ -0.15716 \end{vmatrix}, \quad r=1.0069$$

The fitting value of $X_3^{(0)}$:

$\hat{X}_3^{(0)}$ =[84.0428,84.005,84.5548,85.6634,82.9128,82.7958,75.6862,72.0693,64.2996]

The absolute error of $X_3^{(0)}$:

$q=$ [-0.15716,-0.39504,-1.7452,1.3634,1.6128,7.8958,-0.013812,-1.1307,-2.6004]

The relative error of $X_3^{(0)}$:

$e=$ [-0.1866,-0.4681,-2.0223,1.6174,1.9838,10.5418,-0.01825,-1.5446,-3.8870]

The average relative errors of $X_1^{(0)}$, $X_2^{(0)}$ and $X_3^{(0)}$ are 2.5292 %, 2.3643 % and 2.4744% respectively. The average relative error of this model is 2.456%. Thus, the accuracy of the model is higher. When r = 1 and not optimizing β , their average relative errors are 3.4811%, 3.5508% and 2.0206%, and the average relative error of this model is 3.0175%.

IV. CONCLUSION

(1) The fractional order, the background value and the initial value of the differential equation have all great influence on the performance of the model. Constructing background value based on vector continued fractions theory by using rational interpolation, trapezoidal rule in numerical integration and extrapolation formula, the non-equidistant multi-variable grey model NFMGM(1,n) with the fractional order accumulation was established. Taking the mean relative error as objective function and taking the modified values of response function initial values and the number of fractional order as design variables, multi-variable non-equidistance grey optimizing model ONFMGM(1,n) with the fractional order accumulation was built.

(2) The non-equidistant multi-variable grey model NFMGM(1,n) with the fractional order accumulation can better reveal the nature of the object and its behavior than the non-equidistant multi-variable accumulation grey model NMGM(1,n). This model can reflect the restricted and affected multi-variables relation with one another.

(3) This model can be used to build model in non-equidistant and equidistant time sequences, and has high precision and easy to use. Examples validate the proposed model with the practicability and reliability has important theoretical and practical value.

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