Non-Equidistant Non-Homogeneous Multivariate Grey Model with Fractional Order Accumulation and its Application

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***Abstract* — We consider fitting approximations to non-homogenous series to build non-equidistant multivariate grey model NFMGM (1, n) with fractional order accumulation by fitting data with a homogenous exponential function. There are many approximations to non-homogenous series. Based on the modeling principle of non-equidistant multivariate grey model NFMGM (1, n) with the fractional order accumulation, we propose a non-equidistant non-homogeneous multivariate grey model NNFMGM (1, n) with fractional order accumulation. The parameters of the proposed model are estimated by least square method and the time response function is given. We consider: i) the number of fractional order, ii) the coefficient of the background value, iii) the modified values of response function initial values as design variables, and the iv) minimum average relative error as object function. We then establish the optimal model and write the solution program using Matlab. The resulting model with high precision and adaptability is not only suitable for equidistant modeling, but also for non-equidistant modeling. Examples show that the model is practical and reliable, and has wide potential applications in engineering and related fields.**

Keywords - grey model; fractional order; non-equidistant; non-homogeneous exponential function; least squares method; model parameter estimation

I. INTRODUCTION

Grey model GM (1,1) reveals the inherent development law by first-order differential equation model with single variable which is an integer order derivative model [1], while in fact, there are mostly the non-equidistant data sequences with fractional order in the real system. That using fractional order to describe the object with fractional order property can better reveal their nature and behavior. The grey model FGM (1,1) with fractional order accumulation was proposed [2], and the weapons maintenance fee was predicted using FGM (1,1) [3]. The discrete grey model DFGM (1,1) with fractional order accumulation was proposed [4]. FGM (1,1) and DFGM (1,1) was summarized and the applicable range of FGM (1,1) was researched [5]. The optimal solution of FGM (1,1) was studied [6], but these models all belong to equidistant models. GM (1,1) with single variable was extended to the multivariate grey model MGM (1, n) whose background is generated by the mean [7]. MGM (1,n) is neither the simple combination of GM (1, n), nor GM (1, n) that establishing only a first-order differential equation with n variables. In MGM (1,n), n differential equations with n variables are established and solved, in order that the parameters in the model can reflect the relationships of mutual influence and restriction among multiple variables. The improved method of the background value in a non-equidistant GM (1,1) was proposed [8,9], where the homogeneous exponential function was used to fit the one-time accumulated generating sequence and the higher accuracy is obtained. The non-homogeneous exponential function was used to fit the one-time accumulated generating sequence and the model accuracy is further improved [10]. The MGM (1, n) models with the non-equidistant multivariate and with the multivariate new information were established respectively [11,12]. The non-equidistant grey model NFGM (1,1) with the fractional order accumulation and the non-equidistant multivariate grey model NFMGM (1, n) with the fractional order accumulation were studied [13,14]. The non-homogeneous exponential grey model based on equidistant sequence was established to achieve the desired results [15, 16]. The non-homogeneous exponential grey model with non-equidistant sequence was established [17]. This model has higher precision and wider application range, but it is for GM (1,1) with a single variable. There is higher bias on fitting approximation non-homogenous series for building the non-equidistant multivariate grey model NFMGM (1, n) with the fractional order accumulation by fitting data with homogenous exponential function. In fact, there is a lot of approximation non-homogenous series. Based on the modeling principle of non-equidistant multivariate grey model NFMGM (1, n) with the fractional order accumulation, a non-equidistant non-homogeneous multivariate grey model NNFMGM (1, n) with the fractional order accumulation was put forward. The parameters were estimated of the proposed model by least square method and the time respond function was given. By taking the number of fractional order, the coefficient of the background value and the modified values of response function initial values as design variables and the minimum average relative error as object function, the optimal model was established and the solution program based on Matlab was written. NNFMGM (1, n) becomes a non-equidistant grey model NFMGM (1, n) with the fractional order accumulation when $\mathbf{B}_2 = \mathbf{0}$. NNFMGM (1, n) is the promotion of NFMGM (1, n), and NFMGM (1, n) is a special case of NNFMGM (1, n) when $\mathbf{B}_2 = \mathbf{0}$. This model with important theoretical and practical value widens application of grey prediction theory. The model with high precision and adaptability is not only suitable for equidistant modeling, but also for non-equidistant modeling. Examples show that the model is practical and reliable, and it is worth widely applying in engineering and other related fields.

II. MODELING MECHANISM OF NON-EQUIDISTANT MULTIVARIATE GREY MODEL NNFMGM (1, N) WITH FRACTIONAL ORDER ACCUMULATION

Definition 1: Supposed the non-negative sequence $X_i^{(0)} = [x_i^{(0)}(t_1), \cdots, x_i^{(0)}(t_j), \cdots, x_i^{(0)}(t_m)]$, if $\Delta t_j = t_j - t_{j-1} \neq const$ where $i = 1,2,\cdots,n, j = 2,\cdots,m$, n is the number of variables and m is the sequence number of each variable, $X_i^{(0)}$ is called as non-equidistant sequence.

Definition 2: Supposed the sequence $X_i^{(1)} = [x_i^{(1)}(t_1), \cdots, x_i^{(1)}(t_j), \cdots, x_i^{(1)}(t_m)]$, if $x_i^{(1)}(t_1) = x_i^{(0)}(t_1)$ and $x_i^{(1)}(t_j) = x_i^{(1)}(t_{j-1}) + x_i^{(0)}(t_j) \cdot \Delta t_j$ where $j = 2,\cdots,m, i = 1,2,\cdots,n,$ and $\Delta t_j = t_j - t_{j-1}$, $\mathbf{X}_i^{(1)}$ is first-order accumulated generation of non-equidistant sequence $\mathbf{X}_i^{(0)}$, and it is denoted by 1-AG0.

Definition 3: Supposed the non-negative sequence $X_i^{(0)} = [x_i^{(0)}(t_1), \cdots, x_i^{(0)}(t_j), \cdots, x_i^{(0)}(t_m)]$, if $\Delta t_i = t_i - t_{i-1} \neq const$ where $i = 2, \cdots, m$, m is the sequence number of each variable, $X_i^{(r)} = [x_i^{(r)}(t_1) \ \ x_i^{(r)}(t_2) \ \cdots \ x_i^{(r)}(t_m)]$ is called as r-order accumulated generation of $X_i^{(0)}$, where

$$x_i^{(r)}(t_k) = \sum_{i_1=1}^{k} x_i^{(r-1)}(t_{i_1})\Delta t_{i_1} = \begin{cases} x_i^{(r)}(t_{k-1}) + x_i^{(r-1)}(t_k)\Delta t_k, k = 2,3,\cdots,m \\ x_i^{(r)}(t_1) = x_i^{(0)}(t_1) \qquad k = 1 \end{cases} \tag{1}$$

According to the principle of matrix operations, it is obtained that $x_i^{(r)} = A_1 x_i^{(r-1)} = A_1 A_1 x_i^{(r-2)} = \cdots = A_1^{\mathbf{r}} x_i^{(0)}$ where first-order accumulated generation matrix A_1 is

$$A_1 = \begin{vmatrix} 1 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$$

A_1^r is called as the accumulated generation matrix of r.

Combined with the combination number, r is extended from integer to fraction, and the fractional accumulated generating matrix can be obtained. When r is fraction the following formula can be obtained [14].

$$x_i^{(r)}(\mathrm{k}) = \sum_{i_1=1}^{k} \frac{\Gamma(r+k-1)}{\Gamma(k-i_1+1)\Gamma(r)} x_i^{(0)}(i_1) \tag{2}$$

where, $k = 1,2,\cdots,m$, Γ is Gamma function.

Supposed the multivariate original data matrix is

$$X^{(0)} = \{X_1^{(0)}, X_2^{(0)}, \cdots, X_n^{(0)}\}^{\mathbf{T}} = \begin{bmatrix} x_1^{(0)}(t_1) & x_1^{(0)}(t_2) & \cdots & x_1^{(0)}(t_m) \\ x_2^{(0)}(t_1) & x_2^{(0)}(t_2) & \cdots & x_2^{(0)}(t_m) \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{(0)}(t_1) & x_n^{(0)}(t_2) & \cdots & x_n^{(0)}(t_m) \end{bmatrix} \tag{3}$$

where $X^{(0)}(t_j)(j=1,2,\cdots m)$ is the observed values of the variables at t_j and $[x_i^{(0)}(t_1), \cdots, x_i^{(0)}(t_j), \cdots, x_i^{(0)}(t_m)]$ is the non-equidistant sequence, that is, $t_j - t_{j-1}$ is not constant.

The equations of NFMGM(1,n) are r-order differential equations with n variables.

$$\begin{cases} \dfrac{dx_1^{(r)}}{dt} = a_{11}x_1^{(r)} + a_{12}x_2^{(r)} + \cdots + a_{1n}x_n^{(r)} + b_1 \\ \dfrac{dx_2^{(r)}}{dt} = a_{21}x_1^{(r)} + a_{22}x_2^{(r)} + \cdots + a_{2n}x_n^{(r)} + b_2 \\ \cdots \\ \dfrac{dx_n^{(r)}}{dt} = a_{n1}x_1^{(r)} + a_{n2}x_2^{(r)} + \cdots + a_{nn}x_n^{(r)} + b_n \end{cases} \tag{4}$$

Supposed $X^{(0)}(t_k)=(x_1^{(0)}(t_k),x_2^{(0)}(t_k),\cdots,x_n^{(0)}(t_k))^{\mathrm{T}}$, $X^{(r)}(t_k)=(x_1^{(r)}(t_k),x_2^{(r)}(t_k),\cdots,x_n^{(r)}(t_k))^{\mathrm{T}}$,

$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $B=\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$, Eq.(4) can be expressed as:

$$\frac{dX^{(r)}}{dt}=AX^{(r)}+B \tag{5}$$

Taken the first component $x_i^{(r)}(t_1)$ of the sequence $x_i^{(r)}(t_j)$ where $j=1,2,\cdots,m$ as an initial condition of the grey differential equation, the continuous time response of Eq.(5) is as:

$$X^{(r)}(t)=e^{\mathbf{A}t}X^{(r)}(t_1)+A^{-1}(e^{\mathbf{A}t}-I)B \tag{6}$$

where, $e^{At}=I+\sum_{k=1}^{\infty}\frac{A^k}{k!}t^k, I$ is an unit matrix.

In order to identify $\mathbf{A}$ and $\mathbf{B}$, Eq.(4) is discreted and the following equation can be obtained:

$$x_i^{(r-1)}(t_k)=\sum_{j=1}^{n}\frac{a_{ij}}{2}(x_j^{(r)}(t_k)+x_j^{(r)}(t_{k-1}))+b_i \tag{7}$$

where $i=1,2,\cdots,n$ and $k=2,3,\cdots,m$.

Noting $a_i=\left(a_{i1},a_{i2},\cdots,a_{in},b_i\right)^T$, the identified value $\hat{\mathbf{a}}_i$ of a_i can be obtained through the least square method:

$$\hat{a}_i=[\hat{a}_{i1},\hat{a}_{i2},\cdots,\hat{a}_{in},\hat{b}_i]^T=(Z^TZ)^{-1}L^TY_i \tag{8}$$

where,

$$Z=\begin{bmatrix} \frac{1}{2}(x_1^{(r)}(t_2)+x_1^{(r)}(t_1)) & \frac{1}{2}(x_2^{(r)}(t_2)+x_2^{(r)}(t_1)) & \cdots & \frac{1}{2}(x_n^{(r)}(t_2)+x_n^{(r)}(t_1)) & 1 \\ \frac{1}{2}(x_1^{(r)}(t_3)+x_1^{(r)}(t_2)) & \frac{1}{2}(x_2^{(r)}(t_3)+x_2^{(r)}(t_2)) & \cdots & \frac{1}{2}(x_n^{(r)}(t_3)+x_n^{(r)}(t_2)) & 1 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ \frac{1}{2}(x_1^{(r)}(t_m)+x_1^{(r)}(t_{m-1})) & \frac{1}{2}(x_2^{(r)}(t_m)+x_2^{(r)}(t_{m-1})) & \cdots & \frac{1}{2}(x_n^{(r)}(t_m)+x_n^{(r)}(t_{m-1})) & 1 \end{bmatrix} \tag{9}$$

$Y_i=[x_i^{(r-1)}(t_2),x_i^{(r-1)}(t_3),\cdots,x_i^{(r-1)}(t_m)]^{\mathrm{T}}$

Then the identified values of A and B can be get:

$$\hat{A}=\begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{a}_{n1} & \hat{a}_{n2} & \cdots & \hat{a}_{nn} \end{bmatrix}, \quad B=\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \cdots \\ \hat{b}_n \end{bmatrix} \tag{10}$$

The calculated value in NFMGM(1,n) is:

$$\hat{X}_i^{(r)}(t_j)=e^{\hat{A}(t_j-t_1)}X_i^{(r)}(t_1)+\hat{A}^{-1}(e^{\hat{A}(t_j-t_1)}-I)\hat{B} \tag{11}$$

In the above equation, taking the first component of the original data sequence as an initial condition of the grey differential equation, the fitting values of the original data can be get after restoring where $(A_1)^{-1}=A_1^{-r}, A_1^{-r}A_1^r=I, x^{(0)}=A_1^{-r}A_1^rx^{(0)}=A_1^{-r}x^{(r)}$. Defined A_1^{-r} as the r-order inverse accumulated generating matrix, $\hat{X}_i^{(r)}$ is restored as the calculation formula of the original sequence $\hat{X}_i^{(0)}$ [13,14]:

$$\hat{x}_i^{(0)}(t_k)=\sum_{i_1=0}^{k-1}(-1)^i\frac{\Gamma(r+1)}{\Gamma(i_1+1)\Gamma(r-i_1+1)}\hat{x}_i^{(r)}(t_{k-i})/\Delta t_k \tag{12}$$

The absolute error of the ith variable:

$q_i(t_k)=\hat{x}_i^{(0)}(t_k)-x_i^{(0)}(t_k)$

The relative error of the ith variable (%):

$$e_i(t_k)=\frac{\hat{x}_i^{(0)}(t_k)-x_i^{(0)}(t_k)}{x_i^{(0)}(t_k)}\times 100 \tag{13}$$

The mean of the relative error of the ith variable:

$$\text{MAPE(i)}=\frac{1}{m}\sum_{k=1}^{m}\left|e_i(t_k)\right| \tag{14}$$

The average error of all the data:

$$f=\frac{1}{nm}\sum_{i=1}^{n}(\sum_{k=1}^{m}\left|e_i(t_k)\right|) \tag{15}$$

The above model is a non-equidistant grey model NFMGM (1, n) with the fractional order accumulation. When r=1, it becomes a non-equidistant multivariable grey model NMGM (1,n), and becomes an equidistant grey model FMGM(1,n) with the fractional order accumulation. When the number of variables is 1, it becomes GM (1,1) with a single variable. The solution of the model can be calculated after the fractional order is given.

The background value of the above model is generated by the means, denoted as NFMGM-1. The fractional order r can be obtained by using the optimization method. It is amended as:

$$z_i^{(r)}(t_k)=\lambda x_i^{(r)}(t_k)+(1-\lambda)x_i^{(1)}(t_{k-1}) \tag{16}$$

Taken advantage of known information, the background value can be get:

$$Z=\begin{bmatrix} \lambda x_1^{(r)}(t_2)+(1-\lambda)x_1^{(r)}(t_1) & \lambda x_2^{(r)}(t_2)+(1-\lambda)x_2^{(r)}(t_1) & \cdots \\ \lambda x_1^{(r)}(t_3)+(1-\lambda)x_1^{(r)}(t_2) & \lambda x_2^{(r)}(t_3)+(1-\lambda)x_2^{(r)}(t_2) & \cdots \\ \cdots & \cdots & \cdots \\ \lambda x_1^{(r)}(t_m)+(1-\lambda)x_1^{(r)}(t_{m-1}) & \lambda x_2^{(r)}(t_m)+(1-\lambda)x_2^{(r)}(t_{m-1}) & \cdots \end{bmatrix}$$
$$\begin{bmatrix} \lambda x_n^{(r)}(t_2)+(1-\lambda)x_n^{(r)}(t_1 & 1 \\ \lambda x_n^{(r)}(t_3)+(1-\lambda)x_n^{(r)}(t_2 & 1 \\ \cdots & 1 \\ \lambda x_n^{(r)}(t_m)+(1-\lambda)x_n^{(r)}(t_{m-1} & 1 \end{bmatrix} \tag{17}$$

where $\lambda\in[0,1]$ [7,8].

That Eq.(17) substituting Eq.(9) can obtain λ and fractional order r by using the optimization method. The NFMGM model is referred to as NFMGM-2.

In NFMGM-1, the first column of data is taken as the initial value of the solution $x_i^{(r)}(t_1)=x_i^{(r)}(t_1)$. After it is amended, that $x_i^{(r)}(1)+\beta$ takes the place of $x_i^{(r)}(1)$, where β is a vector whose dimension is equal to $x_i^{(r)}(1)$, that is, $\beta=[\beta_1,\beta_2,\cdots,\beta_n]^T$. Eq.(11) is changed as:

$$\hat{X}_i^{(r)}(t_k)=e^{\hat{A}(t_k-t_1)}(X_i^{(r)}(t_1)+\beta)+\hat{A}^{-1}(e^{\hat{A}(t_k-t_1)}-I)\hat{B} \tag{18}$$

$\hat{X}_i^{(r)}$ is restored as the original sequence $\hat{X}_i^{(0)}$.

That Eq.(17) substituting Eq.(9) and Eq.(18) substituting Eq.(11) can obtain λ, β and fractional order r by using the optimization method. The NFMGM model is referred to as NFMGM-3.

After analyzing Eq.(11), it is found $\hat{X}_i^{(0)}$ has homogeneous exponent characteristic when $\hat{X}_i^{(r)}$ is restored to the original sequence $\hat{X}_i^{(0)}$. Because the collected data in the practical application are often approximate non-homogeneous, the non-homogeneous exponent sequence is used to fit the original data in this paper.

III. Modeling Mechanism of Non-Equidistant Non-Homogeneous Multivariate Grey Model NNFMGM (1, n) with Fractional Order Accumulation

Supposed the non-negative sequence $X_i^{(1)}=[x_i^{(1)}(t_1),\cdots,x_i^{(1)}(t_j),\cdots,x_i^{(1)}(t_m)]$, where $i=1,2,\cdots,n$, $j=2,\cdots,m$, n is the number of variables and m is the sequence number of each variable, $X_i^{(r)}=[x_i^{(r)}(t_1),\cdots,x_i^{(r)}(t_j),\cdots,x_i^{(r)}(t_m)]$ is called as is r-order accumulated generation of $\hat{X}_i^{(0)}$, denoted as r-AG0. Supposed that $Z^{(r)}(t)$ is the background value of $X^{(r)}(t)$, $X^{(r-1)}(t)=AZ^{(r)}(t)+B_1+B_2t$ is defined as the differential equation of the order grey model FMGM (1, n) with the fractional order accumulation under the optimization on the grey action.

Albino differential equations of FMGM (1, n) are r-order differential equations with n variables.

$$\begin{cases} \frac{dx_1^{(1)}}{dt}=a_{11}x_1^{(1)}+a_{12}x_2^{(1)}+\cdots+a_{1n}x_n^{(1)}+b_1+b_{21}t \\ \frac{dx_2^{(1)}}{dt}=a_{21}x_1^{(1)}+a_{22}x_2^{(1)}+\cdots+a_{2n}x_n^{(1)}+b_2+b_{22}t \\ \cdots \\ \frac{dx_n^{(1)}}{dt}=a_{n1}x_1^{(1)}+a_{n2}x_2^{(1)}+\cdots+a_{nn}x_n^{(1)}+b_n+b_{2n}t \end{cases} \tag{19}$$

Noting $X^{(r-1)}(t_k)=(x_1^{(r-1)}(t_k),x_2^{(r-1)}(t_k),\cdots,x_n^{(r-1)}(t_k))^{\mathrm{T}}$, $X^{(r)}(t_k)=(x_1^{(r)}(t_k),x_2^{(r)}(t_k),\cdots,x_n^{(r)}(t_k))^{\mathrm{T}}$, then,

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},\quad B_1=\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix},\quad B_2=\begin{bmatrix} b_{11} \\ b_{22} \\ \cdots \\ b_{nn} \end{bmatrix},$$

Eq.(19) can be expressed as:

$$\frac{dX^{(r)}}{dt}=AX^{(r)}+B_1+B_2t \tag{20}$$

Taken the first component $x_i^{(r)}(t_1)$ of the sequence $x_i^{(r)}(t_j)(j=1,2,\cdots,m)$ as an initial condition of the grey differential equation, the continuous time response of Eq.(20) is as:

$$\begin{aligned}X^{(r)}(t)=&(X^{(r)}(t_1)+A^{-1}B_1+A^{-1}B_2t_1+A^{-2}B_2)e^{A(t-t_1)}\\&-A^{-1}B_1-A^{-1}B_2t-A^{-2}B_2\end{aligned} \tag{21}$$

where, $e^{At}=I+\sum_{k=1}^{\infty}\frac{A^k}{k!}t^k, I$ is an unit matrix.

In order to identify $\mathbf{A}$, B_1 and B_2, Eq.(19) is discretized and $X^{(r-1)}(t)=AZ^{(r)}(t)+B_1+B_2t$ can be obtained through the difference grey derivative in $[t_{k-1},t_k]$. Taken $z_i^{(r)}(t_k)=0.5*(x_i^{(r)}(t_k)+x_i^{(r)}(t_{k-1}))$, the following equation can be obtained:

$$x_i^{(r-1)}(t_k)=\sum_{j=1}^{n}\frac{a_{ij}}{2}(x_j^{(r)}(t_k)+x_j^{(r)}(t_{k-1}))+b_{1i}+b_{2i}t_k \tag{22}$$

where $i=1,2,\cdots,n$ and $k=2,3,\cdots,m$.

The definite integral is taken on both sides of the equation in $[t_{k-1},t_k]$:

$$\begin{aligned}x_i^{(r-1)}(k)\Delta t_k=&\Delta t_k\sum_{j=1}^{n}\frac{a_{ij}}{2}(x_j^{(r)}(t_k)+x_j^{(r)}(t_{k-1}))+\Delta t_k b_{1i}\\&+b_{2i}(t_k^2-t_{k-1}^2)/2\end{aligned} \tag{23}$$

where, $\Delta t_k=t_k-t_{k-1}$.

Noting $a_i=(a_{i1},a_{i2},\cdots,a_{in},b_{1i},b_{2i})^{\mathrm{T}}(i=1,2,\cdots,n)$, the identified value $\hat{\mathbf{a}}_i$ of a_i can be obtained through the least square method:

$$\hat{a}_i=(\hat{a}_{i1},\hat{a}_{i2},\cdots,\hat{a}_{in},\hat{b}_{1i},\hat{b}_{2i})^{\mathrm{T}}=(Z^{\mathrm{T}}Z)^{-1}Z^{\mathrm{T}}Y_i \tag{24}$$

where,

$$Z=\begin{bmatrix} z_1^{(r)}(t_2)\Delta t_2 & z_2^{(r)}(t_2)\Delta t_2 & \cdots & z_n^{(r)}(t_2)\Delta t_2)) & \Delta t_2 & \frac{t_2^2-t_1^2}{2} \\ z_1^{(r)}(t_3)\Delta t_3 & z_2^{(r)}(t_3)\Delta t_3 & \cdots & z_n^{(r)}(t_3)\Delta t_3 & \Delta t_3 & \frac{t_3^2-t_2^2}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ z_1^{(r)}(t_m)\Delta t_m & z_2^{(r)}(t_m)\Delta t_m & \cdots & z_n^{(r)}(t_m)\Delta t_m & \Delta t_m & \frac{t_m^2-t_{m-1}^2}{2} \end{bmatrix} \tag{25}$$

$$Y_i=[x_i^{(r-1)}(t_2)\Delta t_2,x_i^{(r-1)}(t_3)\Delta t_3,\cdots,x_i^{(r-1)}(t_m)\Delta t_m]^{\mathrm{T}} \tag{26}$$

Then the identified values of A, B_1 and B_2 can be get:

$$\hat{A}=\begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{a}_{n1} & \hat{a}_{n2} & \cdots & \hat{a}_{nn} \end{bmatrix},\quad B_1=\begin{bmatrix} \hat{b}_{11} \\ \hat{b}_{12} \\ \cdots \\ \hat{b}_{1n} \end{bmatrix},\quad B_2=\begin{bmatrix} \hat{b}_{21} \\ \hat{b}_{22} \\ \cdots \\ \hat{b}_{2n} \end{bmatrix} \tag{27}$$

The calculated value in FMGM(1,n) is:

$$\begin{aligned}\hat{X}^{(r)}(t_k)=&(X^{(1)}(t_1)+A^{-1}B_1+A^{-1}B_2t_1+A^{-2}B_2)e^{A(t_k-t_1)}\\&-A^{-1}B_1-A^{-1}B_2t_k-A^{-2}B_2\end{aligned} \tag{28}$$

$\hat{X}_i^{(r)}$ is restored to the original sequence $\hat{X}_i^{(0)}$.

After analyzing the above equation, it is found $\hat{X}_i^{(0)}$ has homogeneous exponent characteristic. FMGM (1, n) is referred to NNFMGM(1,n), where the first N represents non-equidistant and the second N represents non-homogeneous exponent.

The absolute error of the ith variable:

$$q_i(t_k)=\hat{x}_i^{(0)}(t_k)-x_i^{(0)}(t_k) \tag{29}$$

The relative error of the ith variable (%):

$$e_i(t_k)=\frac{\hat{x}_i^{(0)}(t_k)-x_i^{(0)}(t_k)}{x_i^{(0)}(t_k)} \tag{30}$$

The mean of the relative error of the ith variable:

$$\text{MAPLE(i)}=\frac{1}{m}\sum_{k=1}^{m}|e_i(t_k)|\times 100 \tag{31}$$

The average error of all the data:

$$f = \frac{1}{nm}\sum_{i=1}^{n}(\sum_{k=1}^{m}|e_i(t_k)|)\times 100 \quad (32)$$

The background value of the above model is generated by the mean, denoted as NNFMGM-1. The fractional order r can be obtained through the optimization method. After amended and taken advantage of known information, the background value can be get:

$$\mathbf{Z}=\begin{bmatrix} z_1^{(r)}(t_2)\Delta t_2 & z_2^{(r)}(t_2)\Delta t_2 & \cdots & z_n^{(r)}(t_2)\Delta t_2) & \Delta t_2 & \frac{t_2^2-t_1^2}{2} \\ z_1^{(r)}(t_3)\Delta t_3 & z_2^{(r)}(t_3)\Delta t_3 & \cdots & z_n^{(r)}(t_3)\Delta t_3 & \Delta t_3 & \frac{t_3^2-t_2^2}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ z_1^{(r)}(t_m)\Delta t_m & z_2^{(r)}(t_m)\Delta t_m & \cdots & z_n^{(r)}(t_m)\Delta t_m & \Delta t_m & \frac{t_m^2-t_{m-1}^2}{2} \end{bmatrix} \quad (33)$$

where，

$z_i^{(r)}(t_k)=\lambda x_i^{(r)}(t_k)+(1-\lambda)x_i^{(r)}(t_{k-1}),\ \lambda\in[0,1]$ [18].

That Eq.(33) substituting Eq.(25) can obtain λ and the fractional order r through the optimization method. The NNMGM model is referred to as NNFMGM-2.

In NNFMGM-1, the first column of data is taken as the initial value of the solution $x_i^{(r)}(t_1)=x_i^{(0)}(t_1)$. After it is amended, that $x_i^{(r)}(t_1)+\beta$ takes the place of $x_i^{(r)}(t_1)$, where β is a vector whose dimension is equal to $x_i^{(r)}(t_1)$, that is, $\beta=[\beta_1,\beta_2,\cdots,\beta_n]^T$. Eq.(28) is changed as:

$$\hat{X}^{(r)}(t_k)=(X^{(r)}(t_1)+\beta+A^{-1}B_1+A^{-1}B_2+A^{-2}B_2)e^{A(t_k-t_1)} -A^{-1}B_1-A^{-1}B_2t_k-A^{-2}B_2 \quad (34)$$

$\hat{X}_i^{(r)}$ is restored to the raw sequence $\hat{X}_i^{(0)}$.

The fractional accumulated generating operation was compiled to the function as RAGO.m, and the inverse accumulated generating operation as IRAGO.m. According to the above method establishing model, the model solution NNFMGM.m was written as the objective function, and then the main main_NNFMGM.m was written. That Eq.(33) substituting Eq.(25) and Eq.(34) substituting Eq.(28) can obtain λ, β and the fractional order r through the optimization method such as genetic optimization function ga.m in MATLAB, elite parent body genetic optimization [19], quantum chaos particle swarm optimization [20] and other modern intelligent optimization method. The NNFMGM model is referred to as NNFMGM-3. After solving the model, the model should be tested to determine whether the model is appropriate.

In this paper, NNFMGM (1, n) becomes a non-equidistant grey model NFMGM (1, n) with the fractional order accumulation when $\mathbf{B}_2=\mathbf{0}$. NNFMGM (1, n) is the promotion of NFMGM (1, n), and NFMGM (1, n) is a special case of NNFMGM (1, n) when $\mathbf{B}_2=\mathbf{0}$. This model with important theoretical and practical value widens application of grey prediction theory.

IV. EXAMPLES

On the lathe CA6140, that using the tool with YT14 lathed the cylindrical. Ensuring that the geometry parameters of the tool and the cutting speed are constant, the cut depth is changing to measure the cutting force as shown in Table 1 [21].

TABLE 1. CUTTING EXPERIMENTAL DATA WHEN F=0.02MM/R

No.	1	2	3	4	5
a_p / mm	1.00	1.25	1.50	1.75	2.00
F_{1z} / N	838.98	1060.45	1261.79	1483.25	1704.72
F_{1y} / N	255.10	290.16	355.22	420.28	469.08

Denoting the cutting depth a_p as t_k, the cutting force F_{1z} as $\hat{X}_1^{(0)}$ and F_{1y} as $\hat{X}_2^{(0)}$, and establishing three models of NFMGM (1,2) and NNFMGM (1,2) respectively according to the method in this paper, the results are shown from Table 2 to Table 5. It is shown that this model is practical and reliable from Table 2 to Table 5.

TABLE 2. COMPARISON OF THE FITTING VALUE AMONG NON-EQUIDISTANT MODELS

t_k	NFMGM-1(1,2)		NFMGM-2(1,2)		NFMGM-3(1,2)	
	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$
1.00	838.98	255.1	838.98	255.1	838.118176	254.23818
1.25	1039.25771	283.44275	1060.44716	289.49245	1056.23565	290.16001
1.50	1265.95221	360.07481	1272.93174	361.85433	1262.83123	358.57073
1.75	1483.2692	416.78114	1496.6792	420.28009	1498.03804	420.27998
2.00	1710.92445	473.85462	1717.80634	475.34529	1697.4927	469.08173

TABLE 3. COMPARISON OF THE ACCURACY AMONG NON-EQUIDISTANT MODELS

Average relative error (%) of NFMGM-1(1,2)		Average relative error (%) of NFMGM-2(1,2)		Average relative error (%) of NFMGM-3(1,2)	
$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$
0.53871	1.1064	0.51126	0.68668	0.40072	0.2563
0.82256		0.59897		0.32851	

TABLE 4. COMPARISON OF THE FITTING VALUE AMONG NON-EQUIDISTANT NON-HOMOGENEOUS MODELS

t_k	NNFMGM-1(1,2)		NNFMGM-2(1,2)		NNFMGM-3(1,2)	
	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$
1.00	838.98	255.1	838.98	255.1	838.66207	254.7821
1.25	1061.218	290.7037	1061.8402	290.1598	1060.5507	289.5856
1.50	1262.4705	356.5235	1262.9889	354.8394	1261.8296	354.2752
1.75	1483.725	423.9677	1484.2008	419.6462	1483.251	419.1245
2.00	1704.6813	479.8334	1704.8317	469.1831	1704.9841	469.0809

TABLE 5. COMPARISON OF THE ACCURACY AMONG NON-EQUIDISTANT NON-HOMOGENEOUS MODELS

Average relative error (%) of NNFMGM-1(1,2)		Average relative error (%) of NNFMGM-2(1,2)		Average relative error (%) of NNFMGM-3(1,2)	
$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$
0.032129	0.74485	0.059353	0.055998	0.013218	0.17274
0.38849		0.057676		0.092978	

Thus, this model has better adaptability and effectiveness, so it is necessary to build this model.

V. CONCLUSION

(1) There is higher bias on fitting approximation non-homogenous series for building the non-equidistant multivariate grey model NFMGM (1, n) with the fractional order accumulation by fitting data with homogenous exponential function. In fact, there is a lot of approximation non-homogenous series.

(2) Based on the modeling principle of non-equidistant multivariate grey model NFMGM (1, n) with the fractional order accumulation, a non-equidistant non-homogeneous multivariate grey model NNFMGM (1, n) with the fractional order accumulation was put forward. The parameters were estimated of the proposed model by least square method and the time respond function was given. By taking the number of fractional order, the coefficient of the background value and the modified values of response function initial values as design variables and the minimum average relative error as object function, the optimal model was established and the solution program based on Matlab was written.

(3) In this paper, NNFMGM (1, n) becomes a non-equidistant grey model NFMGM (1, n) with the fractional order accumulation when $\mathbf{B}_2 = \mathbf{0}$. NNFMGM (1, n) is the promotion of NFMGM (1, n), and NFMGM (1, n) is a special case of NNFMGM (1, n) when $\mathbf{B}_2 = \mathbf{0}$. This model with important theoretical and practical value widens application of grey prediction theory.

(4) The model with high precision and adaptability is not only suitable for equidistant modeling, but also for non-equidistant modeling. Examples show that the model is practical and reliable, and it is worth widely applying in engineering and other related fields.

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