Multi-Product EOQ Model with Multivariate Markov Demand

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***Abstract* — We consider an optimality of ordering quantity for the multi-product inventory system under the situation with multivariate Markov demand. In order to forecast the multiple demands in the inventory system, we propose a new forecasting model with multivariate Markov chains in which the demands are state-dependent, here we name the new model as multivariate Markov demand model (MMDM), and the relationships between demands are mined through the MMDM. Furthermore, a new multi-product inventory system is established by using the MMDM, and the solution of the model is proposed. The numerical example show that MMDM is an effective method for forecasting multiple demands in the inventory system, and the EOQ model with multivariate Markov demand is a feasible approach for optimizing ordering quantity in the multi-product inventory system.**

Keywords-Multi-product inventory; multivariate Markov demand; EOQ model

I. INTRODUCTION

Traditional Markov chain is a strong theoretical tool for data mining and forecasting, but the high-order chain with too many parameters in the model discourages people from using it to solve real problems in the inventory management system. In order to simplify calculation and enhance the practicability of the model, a new high-order Markov chain is proposed by Raftery [1], i.e.

$$p_{ij}^{(n)} = P\{X_{n+m} = j \mid X_m = i\} = \sum_{l=1}^{n} \lambda_l q_{ji_l}$$

where $\sum_{l=1}^{n} \lambda_l = 1, \lambda_l \geq 0$, $Q = (q_{ji})_{m\times m}$ is a nonnegative definite matrix, and the column sum of Q is equal to one. Based on Raftery's model, Ching et al. [2] propose multivariate Markov chains for modeling multiple categorical data sequences. Multivariate Markov model is a new stochastic process, and it has attracted wide attention and application in a field of randomized control. There are two streams of the literature about this research: (a) Papers on the fundamental theoretical research. Ching et al. [3] proposed an higher-order multivariate Markov chain model for forecasting multiple categorical data sequences, which required significantly less parameters than those of the conventional model. Khare and Zhou [4] studied the rates of convergence in multivariate Markov chains with polynomial eigenfunction, and proposed a class of generalized Bernoulli–Laplace processes. Zhu and Ching [5])present a note on high-dimensional Markov chain models, studying the stationary property of the models. Bielecki et al. [6] considered the intricacies of the dependence between components of multivariate Markov chains with weak Markov consistency and weak Markov copulae, and they gave sufficient and necessary conditions for these weak properties in terms of semi-martingale characteristics. Khare and Mukherjee [7] considered the convergence of some multivariate Markov chains with stochastic monotonicity, and their conclusion show that Markov chains are stochastically monotone with respect to an appropriate partial ordering. Nicolau[8] proposed a new model for the multivariate Markov chains whose constraints are completely free, and it is also more precise for estimating parameters. Beare et al. [9] constructed flexible semi-parametric models for stationary multivariate higher-order Markov chains with vine copula. (b) Papers on the practical application. Ching et al. [10] studied construction of stochastic genetic networks based on multivariate Markov chains, and they also proposed an efficient estimation method for obtaining the model parameters. Zhang et al. [11]proposed an optimal control policy for gene expression sequences with multivariate Markov model. Siu et al. [12] considered the application of multivariate Markov chain model for credit rating and credit transition matrices by using the method of linear programming. Zhu and Ching [13]determined newsvendors model with multivariate Markov chain, and their numerical examples demonstrate that the effectiveness of the proposed methods. Chevallier [14]considered the problem about global imbalances, cross-market linkages, and the financial crisis with multivariate Markov model. Hagen et al. [15]determined a multivariate Markov weather model to present sea state time series based

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on observed time series. Dimitriou et al. [16]utilized the Multivariate Markov chains in manpower planning modeling, which allowed decision-maker to distinguish employees' mobility patterns.

We know that demand state is not equivalent to demand, and they are totally two different concepts. In fact, the former is a qualitative variable and the latter is a quantitative variable, respectively. Ching et al. (as in [2]) established a multivariate Markov model for forecasting multi-product demand states, and the forecasting result in their numerical example turns out that there is noticeable improvement in prediction accuracy, yet the result of prediction is the demand state in the data sequence, not the demand of product. The classical economic order quantity (EOQ) model, however, is based on the result of forecasting demand, rather than the demand state. Therefore, their forecasting result cannot be used in the classical inventory model. Hence, we need some other mathematical tools to transform demand state into demand so as to employ multivariate Markov model in multiple products system.

Multi-product economic order quantity (EOQ) model is a fundamental theory in the inventory control, and it has been wisely used in operations management. Pasandideh et al.[17]considered a genetic algorithm for vendor managed inventory control system of multi-product and multi-constraint EOQ model, and the total inventory cost of the supply chain is minimized by using the optimal ordering quantity derive from the model. Pal et al. [18] studied a multi-item deterministic EOQ model in which the demand rate is dependent on sales price, and the optimal order quantities, optimal selling prices and optimal level of price break are found out to maximize the average profit in multiple products inventory system. Bajaj et al. [19] considered a multi-product EOQ model with quantity discount incorporating partial or full truckload policy and Mousavi et al. [20]studied a multi-product multi-period inventory control problem under the situation with inflation and discount. Nia et al. [21] optimized the multi-product EOQ model with shortage for a single-buyer single-supplier supply chain under green vendor managed inventory policy and the goal of minimizing the total cost of the supply chain is achieved by hybrid genetic and imperialist competitive algorithm.

The optimal order quantity problem in multi-product inventory system has caused extensive attention in the academia, but most of the above models are proposed by assuming that the multiple demands are mutually independent, and little attention has been paid to the inventory with mutually dependent multiple products demands. However, more and more demands are correlated to each other in reality. For example, we suppose that there are competitive or substitutable products A, B and C in the inventory system. Because of the instability of customers' preferences on the products, one may buy product A in the current period and then may purchase product B or C in the next period with her preference is changed by time. It means that these demands can transfer to each other so that the system is actually with the dependent multi-product demands. Under the situation with the mutually dependent demands, apparently, the ordering decision for the inventory is different from the optimal ordering quantity which is presented by the traditional EOQ model, so we need to develop a new method to mine the relationships among the multiple demands and forecast the demands in this inventory system, as well as characterize the optimal decision with the dependent demands. In this paper, we propose a multi-product model with multivariate Markov demand, and find out the optimal ordering quantity to maximize the expected profit in multiple products inventory system.

II. THE MULTIVARIATE MARKOV CHAINS FOR FORECASTING MULTI-DEMAND IN THE MARKET

There are more and more data sequences generated by similar sources to be correlated to each other in the real world, and how to explore these relationships and predict its tendency become challenging problems in the Big Data Era. As a new theoretical tool for forecasting, multivariate Markov chains has perfectly handled this problem.

A. The Multivariate Markov Chains

In this subsection, we introduce multivariate Markov chains to search the relationships between the multiple categorical sequences generated by similar sources. Here we assume that there are N categorical sequences and each sequence has l possible states in the set $\boldsymbol{I} = \{i_1, i_2, \cdots, i_l\}$.

Definition 1 Let $\boldsymbol{P}^{(ji)}$ be a transition probability matrix from the states in the i th sequence to the states in the j th sequence, and $\boldsymbol{X}_k^{(i)}$ is the state probability distribution of the i th sequence at time k. If the following equation is hold, that is

$$\boldsymbol{X}_{k+1}^{(j)} = \sum_{i=1}^{N} \lambda_{ji} \boldsymbol{P}^{(ji)} \boldsymbol{X}_k^{(i)} \text{, for } j = 1,2,\cdots,N \tag{1}$$

where $\sum_{i=1}^{N} \lambda_{ji} = 1, \lambda_{ji} \geq 0$. Then, we name the equation (1) multivariate Markov chains[22].

Setting $\boldsymbol{X}_{k+1} = \left(\boldsymbol{X}_{k+1}^{(1)}, \boldsymbol{X}_{k+1}^{(2)}, \cdots, \boldsymbol{X}_{k+1}^{(N)}\right)^T$, then the multivariate Markov chains can be written in the form of a matrix, that is

$$\boldsymbol{X}_{k+1} = \begin{pmatrix} \boldsymbol{X}_{k+1}^{(1)} \\ \boldsymbol{X}_{k+1}^{(2)} \\ \vdots \\ \boldsymbol{X}_{k+1}^{(N)} \end{pmatrix} = \begin{pmatrix} \lambda_{11}\boldsymbol{P}^{(11)} & \lambda_{12}\boldsymbol{P}^{(12)} & \cdots & \lambda_{1N}\boldsymbol{P}^{(1N)} \\ \lambda_{21}\boldsymbol{P}^{(21)} & \lambda_{22}\boldsymbol{P}^{(22)} & \cdots & \lambda_{2N}\boldsymbol{P}^{(2N)} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{N1}\boldsymbol{P}^{(N1)} & \lambda_{N2}\boldsymbol{P}^{(N2)} & \cdots & \lambda_{NN}\boldsymbol{P}^{(NN)} \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_k^{(1)} \\ \boldsymbol{X}_k^{(2)} \\ \vdots \\ \boldsymbol{X}_k^{(N)} \end{pmatrix} \triangleq \boldsymbol{A}\boldsymbol{X}_k \tag{2}$$

Given the data sequence, let $H_n = \{s_k^{(n)}\}_{k=1}^{K}$ be historical data for the nth sequence. Therefore, we can count the model parameters by calculating the transition frequency $f_{i_j i_i}^{(ji)}$ from the state i_i in the sequence $\{X_k^{(i)}\}$ to the state i_j

in the sequence $\{X_k^{(j)}\}$, thus the transition frequency matrix is constructed as follows:

$$F_k^{(ji)} = \begin{pmatrix} f_{11}^{(ji)} & f_{12}^{(ji)} & \cdots & f_{1l}^{(ji)} \\ f_{21}^{(ji)} & f_{22}^{(ji)} & \cdots & f_{2l}^{(ji)} \\ \vdots & \vdots & \vdots & \vdots \\ f_{l1}^{(ji)} & f_{l2}^{(ji)} & \cdots & f_{ll}^{(ji)} \end{pmatrix} \quad (3)$$

And from the formula (3), we get the estimation for the transition probability matrix $\boldsymbol{P}^{(ji)}$, which is

$$\hat{\boldsymbol{P}}^{(ji)} = \begin{pmatrix} \hat{p}_{11}^{(ji)} & \hat{p}_{12}^{(ji)} & \cdots & \hat{p}_{1l}^{(ji)} \\ \hat{p}_{21}^{(ji)} & \hat{p}_{22}^{(ji)} & \cdots & \hat{p}_{2l}^{(ji)} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{p}_{l1}^{(ji)} & \hat{p}_{l2}^{(ji)} & \cdots & \hat{p}_{ll}^{(ji)} \end{pmatrix}$$

(4)

where $$\hat{p}_{i_j i_i}^{(ji)} = \begin{cases} \dfrac{f_{i_j i_i}^{(ji)}}{\sum_{i_j=1}^{l} f_{i_j i_i}^{(ji)}}, & \text{if } \sum_{i_j=1}^{l} f_{i_j i_i}^{(ji)} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that the transition matrices $\boldsymbol{P}^{(ji)}$ $(1 \le i, j \le N)$ are irreducible and $\lambda_{ji} > 0$, for $1 \le i, j \le N$. By the proposition 3 in the literature [2], we know that there is a unique vector $\boldsymbol{X} = \left(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}, \cdots, \boldsymbol{X}^{(N)}\right)^T$ such that $\lim_{k \to \infty} \boldsymbol{X}_k = \boldsymbol{X}$. Hence, from the equation (2), we have $\boldsymbol{X} = \boldsymbol{A}\boldsymbol{X}$, and the proposition 7.2 in the literature [22] shows that the vector $\boldsymbol{X}$ can be estimated by counting the proportion of the occurrence of each state in the historical data for each sequence.

Denoting the estimation of $\boldsymbol{X}$ by $\hat{\boldsymbol{X}} = \left(\hat{X}^{(1)}, \hat{X}^{(2)}, \cdots, \hat{X}^{(N)}\right)^T$, and letting $\lambda = \{\lambda_{ji}\}$, by the formula (7.5) in the literature [22], we can estimate the parameters λ by solving the following minimization problem:

$$\begin{cases} \min\limits_{\lambda} \max\limits_{m} \left\{ \left| \left[\sum_{i=1}^{l} \lambda_{ji} \hat{\boldsymbol{P}}^{(ji)} \hat{\boldsymbol{X}}^{(i)} - \hat{\boldsymbol{X}}^{(j)} \right]_m \right| \right\} \\ \text{s.t.} \quad \sum_{i=1}^{N} \lambda_{ji} = 1, \\ \qquad \lambda_{ji} \ge 0, \ \forall i, j. \end{cases} \quad (5)$$

Denoting the estimation of λ_{ji} by $\hat{\lambda}_{ji}$, we can write it in the form of a matrix as follows:

$$\hat{\Lambda} = \begin{pmatrix} \hat{\lambda}_{11} & \hat{\lambda}_{12} & \cdots & \hat{\lambda}_{1N} \\ \hat{\lambda}_{21} & \hat{\lambda}_{22} & \cdots & \hat{\lambda}_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\lambda}_{N1} & \hat{\lambda}_{N2} & \cdots & \hat{\lambda}_{NN} \end{pmatrix} \quad (6)$$

Therefore, substituting the variables in equation (2) for formulas (4) and (6), we can obtain that:

$$\hat{\boldsymbol{X}}_{k+1} = \begin{pmatrix} \hat{\lambda}_{11}\hat{\boldsymbol{P}}^{(11)} & \hat{\lambda}_{12}\hat{\boldsymbol{P}}^{(12)} & \cdots & \hat{\lambda}_{1N}\hat{\boldsymbol{P}}^{(1N)} \\ \hat{\lambda}_{21}\hat{\boldsymbol{P}}^{(21)} & \hat{\lambda}_{22}\hat{\boldsymbol{P}}^{(22)} & \cdots & \hat{\lambda}_{2N}\hat{\boldsymbol{P}}^{(2N)} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\lambda}_{N1}\hat{\boldsymbol{P}}^{(N1)} & \hat{\lambda}_{N2}\hat{\boldsymbol{P}}^{(N2)} & \cdots & \hat{\lambda}_{NN}\hat{\boldsymbol{P}}^{(NN)} \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_k^{(1)} \\ \boldsymbol{X}_k^{(2)} \\ \vdots \\ \boldsymbol{X}_k^{(N)} \end{pmatrix} = \hat{\boldsymbol{A}}\hat{\boldsymbol{X}}_k \quad (7)$$

where $$\hat{\boldsymbol{A}} = \begin{pmatrix} \hat{\lambda}_{11}\hat{\boldsymbol{P}}^{(11)} & \hat{\lambda}_{12}\hat{\boldsymbol{P}}^{(12)} & \cdots & \hat{\lambda}_{1N}\hat{\boldsymbol{P}}^{(1N)} \\ \hat{\lambda}_{21}\hat{\boldsymbol{P}}^{(21)} & \hat{\lambda}_{22}\hat{\boldsymbol{P}}^{(22)} & \cdots & \hat{\lambda}_{2N}\hat{\boldsymbol{P}}^{(2N)} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\lambda}_{N1}\hat{\boldsymbol{P}}^{(N1)} & \hat{\lambda}_{N2}\hat{\boldsymbol{P}}^{(N2)} & \cdots & \hat{\lambda}_{NN}\hat{\boldsymbol{P}}^{(NN)} \end{pmatrix}$$

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B. Forecasting Multi-Demand in the Market

In order to forecast multi-demand in the market, here we assume that the multiple demands are state-dependent, and there are N demand sequences and each sequence has l possible states in the set $\boldsymbol{I} = \{i_1, i_2, \cdots, i_l\}$. Let $k = 0, 1, \cdots, K$ and $n = 1, 2, \cdots, N$ be the period and quantity of the products of the inventory problem, so that $d_{nk} \in \boldsymbol{I}$ is the demand state of product n in period k, where $k = 0, 1, \cdots, K$ and $n = 1, 2, \cdots, N$; $\boldsymbol{P}^{(ji)}$ is a transition probability matrix from the states in the i th sequence to the states in the j th sequence, where $\boldsymbol{P}^{(ji)} = (p_{ji})_{l \times l}$, $i, j = 1, 2, \cdots, N$, and p_{ji} is a transition probability from the states in the i th sequence to the states in the j th sequence; ξ_{nk} is demand of product n in period k.

Letting ξ_{nk} to be the demand of product n in period k, we denote that $\phi_{i_t}(\xi_{nk})$ is the probability density function (PDF) of demand ξ_{nk} when $d_{nk} = i_t$, $i_t \in \boldsymbol{I}$, $t = 1, 2, \cdots, l$, and $\boldsymbol{X}_k^{(n)} = (x_{i_1 k}^{(n)}, x_{i_2 k}^{(n)}, \cdots, x_{i_l k}^{(n)})$ is the state probability distribution of the product n at time k. Thus, we can obtain the expected demand of product n at time k, which is:

$$E(\xi_{nk}) = \sum_{t=1}^{l} x_{i_t k}^{(n)} \int_0^{+\infty} \xi_{nk} \phi_{i_t}(\xi_{nk}) d\xi_{nk} \quad (8)$$

In fact, since $\phi_{i_t}(\xi_{nk})$ is the PDF of demand ξ_{nk}, the expected demand of product n is $\int_0^{+\infty} \xi_{nk} \phi_{i_t}(\xi_{nk}) d\xi_{nk}$ when the demand of product n in the state i_t at time k. From the definition of the variable $\boldsymbol{X}_k^{(n)} = (x_{i_1 k}^{(n)}, x_{i_2 k}^{(n)}, \cdots, x_{i_l k}^{(n)})$, we know that the probability value of the system in the demand state i_t at time k is equal to $x_{i_t k}^{(n)}$, $t = 1, 2, \cdots, l$, that is $P(X_k^{(n)} = i_t) = x_{i_t k}^{(n)}$. Hence, we have the equation (8) when the demand state i_t get all the assignments in the state set $\boldsymbol{I}$.

By the Equations (2) and (8), we will develop a new forecasting model for the multi-product inventory system as follows.

The expected demand of various products at time k+1, that is

$$E(\boldsymbol{\xi}_{k+1}) = \boldsymbol{X}_{k+1}\boldsymbol{D}_{k+1} = \boldsymbol{A}\boldsymbol{X}_k\boldsymbol{D}_{k+1} \quad (9)$$

Where

$\boldsymbol{\xi}_{k+1} = (\xi_{1(k+1)}, \xi_{2(k+1)}, \cdots, \xi_{N(k+1)})^T$, $\boldsymbol{D}_{k+1} = (\boldsymbol{D}_{1(k+1)}, \boldsymbol{D}_{2(k+1)}, \cdots, \boldsymbol{D}_{N(k+1)})$, $\boldsymbol{D}_{n(k+1)} = (D_{i_1(k+1)}^{(n)}, D_{i_2(k+1)}^{(n)}, \cdots, D_{i_l(k+1)}^{(n)})^T$, and

$D_{i_t(k+1)}^{(n)} = \int_0^{+\infty} \xi_{n(k+1)} \phi_{i_t}(\xi_{n(k+1)}) d\xi_{n(k+1)}$, $t = 1, 2, \cdots, l;\ n = 1, 2, \cdots, N.$ Here, we name the equation (9) a multivariate Markov demand model.

From the Equation (9), we have

$$E(\xi_{n(k+1)}) = \boldsymbol{X}_{k+1}^{(n)}\boldsymbol{D}_{n(k+1)} = \sum_{m=1}^{N} \lambda_{nm}\boldsymbol{P}^{(nm)}\boldsymbol{X}_k^{(m)}\boldsymbol{D}_{n(k+1)} \quad (10)$$

i.e., the expected demand of product n at time k+1. There are some advantages on forecasting multi-demand in the market by the multivariate Markov demand model. e.g., the multiple demands are forecasted by the equation (9), and the relationships between the different demands are measured by the equation (10). In fact, because $\boldsymbol{X}_k^{(m)}$ is the state probability distribution of the product m at time k, and the relationships between products n and m are determined by the weight coefficient λ_{nm}.

III. MATHEMATICAL MODEL

A. Assumption and Notations

To determine a new EOQ model for the multi-product inventory system, we made the following assumptions and notions for the model.

$k = 0, 1, \cdots, K-1$, the period of the inventory problem, given products $n = 1, 2, \cdots, N$.

T_{nk} = Cycle time of product n in period k.

$\boldsymbol{I} = \{i_1, i_2, \cdots, i_l\}$, a finite set of possible demand states for each product.

$d_{nk} \in \boldsymbol{I}$, the demand state of product n in period k, where $n = 1, 2, \cdots, N$.

ξ_{nk} = Demand of product n in period k.

D_{nk} = Demand rate of product n in period k.

$\boldsymbol{P}^{(ji)}$ = A transition probability matrix from the states in the i th sequence to the states in the j th sequence, where $\boldsymbol{P}^{(ji)} = (p_{ji})_{l \times l}, i, j = 1, 2, \cdots, N$, and p_{ji} is a transition probability from the states in the i th sequence to the states in the j th sequence.

C_{nk} = Replacement cost of product n in period k.

t = Length of time spent in inventory.

L_k = Length of periodic time in the inventory system.

$I_{nk}(t)$ = On-hand inventory level of product n in period k.

Q_{nk} = Ordering quantity of product n in period k.

HC_{nk} = Holding cost per cycle for product n in period k.

TC_{nk} = Total relevant inventory cost per unit time for product n in period k.

TC_k = Total cost of all products in period k; so we have $TC_k = \sum_{n=1}^{N} TC_{nk}$.

Some necessary assumptions for modeling are presented as follows:

A1. Demand ξ_{nk} ($n = 1, \cdots, N$) is a stochastic variable, and its Markovian property and homogeneity of time are hold.

A2. Probability density function (PDF) of demand ξ_{nk} is state-dependent, i.e., $\phi_{i_t}(\xi_{nk})$ is the PDF of demand ξ_{nk} when $d_{nk} = i_t,\ i_t \in \boldsymbol{I},\ t = 1, 2, \cdots, l$.

A3. No backorders are allowed.

A4. Instantaneous replenishment with nonlinear time dependent holding cost.

A5. Let α_n be the shape parameter measures the inventory elasticity, and D_{nk} denotes the scale parameter. The inventory level $I_{nk}(t)$ over the cycle time is given by the following differential equation:

$$\frac{dI_{nk}(t)}{dt} = -D_{nk}I_{nk}^{\alpha_n}(t),\ 0 < \alpha_n < 1,\ 0 \le I_{nk}(t) \le Q_{nk}, \quad (11)$$

with initial condition $I_{nk}(0) = Q_{nk}$.

Solving the differential equation, we obtain that

$$I_{nk}^{1-\alpha_n}(t) = -D_{nk}(1-\alpha_n)t + Q_{nk}^{1-\alpha_n},\ 0 \le t \le T_{nk}. \quad (12)$$

Since $I_{nk}(T_{nk}) = 0$, we get the following equation

$$T_{nk} = Q^{1-\alpha_n} / D_{nk}(1-\alpha_n). \quad (13)$$

B. *Expected Total Cost Model with Multivariate Markov Demand*

By the multivariate Markov demand model, we draw that the expected demand of product n at time k+1 is $E(\xi_{n(k+1)}) = \sum_{m=1}^{N} \lambda_{nm} \boldsymbol{P}^{(nm)} \boldsymbol{X}_{k}^{(m)} \boldsymbol{D}_{n(k+1)}$. Therefore, the demand rate of product n in period k+1 is:

$$D_{n(k+1)} = E(\xi_{n(k+1)}) / L_{k+1} = \sum_{m=1}^{N} \lambda_{nm} \boldsymbol{P}^{(nm)} \boldsymbol{X}_{k}^{(m)} \boldsymbol{D}_{n(k+1)} / L_{k+1} \tag{14}$$

Here we assume that the cost on holding an item $dI_{n(k+1)}(t)$ up is given by $h_n t^m dI_{n(k+1)}(t)$, where $m = 2,3,4,\cdots;\ h_n > 0;\ n = 1,2,\cdots,N$ (see [23]). Thus, we have the holding cost per cycle for product n in period k, that is

$$HC_{n(k+1)} = \int_0^{Q_{n(k+1)}} h_n t^m dI_{n(k+1)}(t), \tag{15}$$

and by the formulations (10), (12) and (14), setting $\beta_n = 1-\alpha_n$, we have

$$HC_{n(k+1)} = D_{n(k+1)} h_n Q_{n(k+1)}^{\alpha_n} \int_0^{T_{n(k+1)}} t^m \left(1 - \frac{D_{n(k+1)}(1-\alpha_n)t}{Q_{n(k+1)}^{1-\alpha_n}} \right)^{\alpha_n/1-\alpha_n} dt,$$

$$= \frac{h_n Q_{n(k+1)}^{m\beta_n+1}}{D_{n(k+1)}^{m} \beta_n^{m+1}} B\left(m+1,\ \frac{1}{\beta_n} \right), \tag{16}$$

where $D_{n(k+1)} = \sum_{m=1}^{N} \lambda_{nm} \boldsymbol{P}^{(nm)} \boldsymbol{X}_{k}^{(m)} \boldsymbol{D}_{n(k+1)} / L_{k+1}$, and $B\left(m+1,\ \frac{1}{\beta_n} \right)$ is a Beta function.

Noting that $\boldsymbol{Q}_{k+1} = \left(Q_{1(k+1)}, Q_{2(k+1)}, \cdots, Q_{N(k+1)} \right)^T$, where $Q_{n(k+1)}$ is the ordering quantity for the product n at period k+1, we consider the multi-product inventory system with no shortage. Then we can obtain the total relevant cost per unit time in period k+1 for the product n as follows:

$$TC_{n(k+1)}(\boldsymbol{Q}_{k+1}) = \frac{C_{n(k+1)} + HC_{n(k+1)}}{T_{nk}} \tag{17}$$

The total cost per unit time for all products in period k+1 is:

$$TC_{k+1}(\boldsymbol{Q}_{k+1}) = \sum_{n=1}^{N} TC_{n(k+1)}(\boldsymbol{Q}_{k+1}) \tag{18}$$

The total cost $TC_{k+1}(\boldsymbol{Q}_{k+1})$ in the new model is different from that of the classical multi-product inventory models, which were mostly constructed with the assumption that the demands are independent[24, 25]. Our model with multivariate Markov demand is hold both for the situations with independent and dependent demand. In fact, the weight coefficient λ_{nm} in the equation (10) reflects the independence between the products. The demands of products n and m are independent when $\lambda_{nm} = 0$, conversely, they are dependent.

C. *Optimal Policies for Multi-product Inventory System*

In this subsection, we construct the optimal policies for multi-product inventory control system in which the multivariate Markov demand is considered.

Let $\{1,2,\cdots,k\}$ be the historical periods of the inventory system, so the next period is k+1. If $\boldsymbol{Q}_{k+1}^{*}$ is the optimal solution for the model, name it the optimal ordering policy for the multi-product inventory system. To find a optimal policy $\boldsymbol{Q}_{k+1}^{*} = \left(Q_{1(k+1)}^{*}, Q_{2(k+1)}^{*}, \cdots, Q_{N(k+1)}^{*} \right)^T$ which satisfies $TC_{k+1}(\boldsymbol{Q}_{k+1}^{*}) = \min_{\boldsymbol{Q}_{k+1}} \{TC_{k+1}(\boldsymbol{Q}_{k+1})\}$, let $\frac{\partial TC_{k+1}(\boldsymbol{Q}_{k+1})}{\partial Q_{n(k+1)}} = 0$, thus we can obtain the optimal ordering quantity for the inventory system, that is

$$Q_{n(k+1)}^{*} = \left(\frac{C_{n(k+1)} D_{n(k+1)}^{m} \beta_n^{m+2}}{h_n (m\beta_n + \alpha_n) B\left(m+1,\ 1/\beta_n\right)} \right)^{1/(m\beta_n+1)} \tag{19}$$

where $D_{n(k+1)} = \sum_{m=1}^{N} \lambda_{nm} \boldsymbol{P}^{(nm)} \boldsymbol{X}_{k}^{(m)} \boldsymbol{D}_{n(k+1)} / L_{k+1}$. The same result is derived in Goh (1994), but the difference is that our optimal policy is constructed by the model with the multivariate Markov demand.

IV. NUMERICAL EXAMPLE

For simplicity, we only consider three products in this paper. We suppose that a company (manufacturer or retailer) sells three products (products 1, 2 and 3) to customers. According to the demand fluctuation in the inventory system, we divide the demand into four states, that is, $\boldsymbol{I} = \{1,2,3,4\}$. We also assume that demand distribution function in the future period k+1 for product n is $\xi_{n(k+1)} \sim U(200(i-1), 200i),\ n = 1,2,3;\ i \in \boldsymbol{I}$, i.e., the probability density function (PDF) of demand in the period k+1 for the product n is

$$\phi_{i,k+1}(\xi_{n(k+1)}) = \begin{cases} \frac{1}{200}, \xi_{n(k+1)} \in [200(i-1), 200i); \\ 0 \quad , \text{otherwise.} \end{cases} \tag{20}$$

where the demand state $d_{n(k+1)} = i$. Thus, by the definition of $\boldsymbol{D}_{k+1}$ in the multivariate Markov demand model, we have:

$$\boldsymbol{D}_{n(k+1)} = (100,300,500,700)^T, n = 1,2,3 \qquad (21)$$

In the following subsection, to explore the relationships among various demands, we will work out correlation coefficient matrix under the situations with the known PDF, and the demands in the system are forecasted by the multivariate Markov demand model. Furthermore, the optimal policies are proposed by the forecasting result of the demands.

A. Correlation Coefficient matrix and Expected Demand

Suppose that the demand state of each product in the historical data is:

$H_1 = \{4,3,1,3,4,4,3,3,1,2,3,4\}$, $H_2 = \{1,2,3,4,1,4,4,3,3,1,3,1\}$

and $H_3 = \{2,1,3,3,2,4,2,3,4,1,4,3\}$, respectively, and transition process that from the demand states in the i th sequence to the states in the j th sequence is illustrated in the following figure.

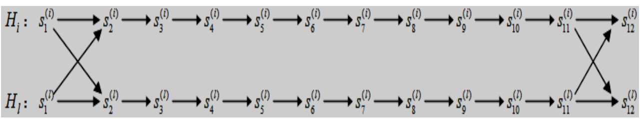

Figure 1. The Demand State Transition.

where $s_k^{(i)}$ and $s_k^{(l)}$ ($i,l = 1,2,3$; $i \neq l$; $1 \le k \le 12$) are the demand states of the product i and product l in the period k, respectively. According to the state transition process show in the Fig.1, we can estimate the transition frequency matrix and probability matrix for the various products shown as follows:

$$\boldsymbol{F}^{(11)} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix} \quad \boldsymbol{F}^{(12)} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad \boldsymbol{F}^{(21)} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

$$\boldsymbol{F}^{(22)} = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad \boldsymbol{F}^{(13)} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \boldsymbol{F}^{(31)} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\boldsymbol{F}^{(23)} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \boldsymbol{F}^{(32)} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix} \quad \boldsymbol{F}^{(33)} = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{P}^{(11)} = \begin{pmatrix} 0 & 0 & 2/5 & 0 \\ 1/2 & 0 & 0 & 0 \\ 1/2 & 1 & 1/5 & 2/3 \\ 0 & 0 & 2/5 & 1/3 \end{pmatrix} \quad \boldsymbol{P}^{(12)} = \begin{pmatrix} 0 & 1 & 1/4 & 0 \\ 0 & 0 & 1/4 & 0 \\ 2/3 & 0 & 1/4 & 2/3 \\ 1/3 & 0 & 1/4 & 1/3 \end{pmatrix} \quad \boldsymbol{P}^{(13)} = \begin{pmatrix} 1/2 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \\ 1/2 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$\boldsymbol{P}^{(21)} = \begin{pmatrix} 1/2 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 1/3 \\ 0 & 1 & 3/5 & 0 \\ 1/2 & 0 & 0 & 2/3 \end{pmatrix} \quad \boldsymbol{P}^{(22)} = \begin{pmatrix} 0 & 0 & 1/2 & 1/3 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 1/4 & 1/3 \\ 1/3 & 0 & 1/4 & 1/3 \end{pmatrix} \quad \boldsymbol{P}^{(23)} = \begin{pmatrix} 0 & 0 & 1/3 & 2/3 \\ 0 & 1/3 & 0 & 0 \\ 1 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$\boldsymbol{P}^{(31)} = \begin{pmatrix} 1/2 & 0 & 0 & 1/3 \\ 0 & 0 & 1/5 & 1/3 \\ 1/2 & 0 & 3/5 & 0 \\ 0 & 1 & 1/5 & 1/3 \end{pmatrix} \quad \boldsymbol{P}^{(32)} = \begin{pmatrix} 1/3 & 0 & 2/4 & 0 \\ 0 & 0 & 0 & 2/3 \\ 0 & 1 & 1/4 & 1/3 \\ 2/3 & 0 & 1/4 & 0 \end{pmatrix} \quad \boldsymbol{P}^{(33)} = \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1/3 & 1/3 \\ 1/2 & 1/3 & 1/3 & 0 \\ 1/2 & 1/3 & 1/3 & 0 \end{pmatrix}$$

where $\boldsymbol{F}^{(ji)}$ and $\boldsymbol{P}^{(ji)}$ are the transition frequency matrix and probability matrix from the states in the i th sequence to the states in the jth sequence, respectively.

Let $t_0 = 12$ be the initial time in the system, then the next period time t_1 is 13. By computing the proportion for the occurrence of each demand state in each of these sequences, we have a probability distribution vector for each product in the system at initial time $t_0 = 12$, that is:

$$\boldsymbol{X}_{12}^{(1)} = (0.167, 0.083, 0.417, 0.333)^T;$$
$$\boldsymbol{X}_{12}^{(2)} = (0.333, 0.083, 0.333, 0.250)^T;$$
$$\boldsymbol{X}_{12}^{(3)} = (0.167, 0.250, 0.333, 0.250)^T.$$

Plugging the matrix $\boldsymbol{F}^{(ji)}$, $\boldsymbol{P}^{(ji)}$ and these probability distribution vectors into the Equation (5), we can obtain a weight coefficient matrix for the multi-product inventory system, that is:

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} = \begin{pmatrix} 0.800 & 0.100 & 0.100 \\ 0.100 & 0.100 & 0.800 \\ 0.502 & 0.100 & 0.398 \end{pmatrix} \qquad (22)$$

As shown in the weight coefficient matrix, we conclude that the demand of product 1 is strongly related with itself with correlation coefficient 0.800. From the matrix, we can also learn that the demand of products 2 and 3 are strongly correlated, and their weight coefficient is 0.800. Similarly, we can obtain the mutual relationships among products by the matrix, e.g., the weight coefficient between product 1 and 3 is 0.502. Hence, the mutual relationships in the multi-product inventory system are revealed by the multivariate Markov model.

Plugging $\boldsymbol{F}^{(ji)}$, $\boldsymbol{P}^{(ji)}$ and Λ into the Equation (7) ,we can obtain the state probability distribution at time 13, that is

$$\boldsymbol{X}_{13} = \begin{pmatrix} 0.800\hat{\boldsymbol{P}}^{(11)} & 0.100\hat{\boldsymbol{P}}^{(12)} & 0.100\hat{\boldsymbol{P}}^{(13)} \\ 0.100\hat{\boldsymbol{P}}^{(21)} & 0.100\hat{\boldsymbol{P}}^{(22)} & 0.800\hat{\boldsymbol{P}}^{(23)} \\ 0.502\hat{\boldsymbol{P}}^{(31)} & 0.100\hat{\boldsymbol{P}}^{(32)} & 0.398\hat{\boldsymbol{P}}^{(33)} \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_{12}^{(1)} \\ \boldsymbol{X}_{12}^{(2)} \\ \boldsymbol{X}_{12}^{(3)} \end{pmatrix}$$
$$= \begin{pmatrix} 0.033 & 0 & 0.633 & 0.333 \\ 0.267 & 0.067 & 0.300 & 0.367 \\ 0.201 & 0.300 & 0.133 & 0.367 \end{pmatrix} \quad (23)$$

Thus, we can gain the probability distribution of state and expected demand in the following Table1.

TABLE 1. THE PROBABILITY DISTRIBUTION FOR STATE I AND EXPECTED DEMAND $\eta_{n,13}(n=1,2,3)$ AT TIME 13.

I	1	2	3	4
$\eta_{n,13}$	75	225	375	525
$\boldsymbol{X}_{13}^{(1)}$	0.033	0	0.633	0.333
$\boldsymbol{X}_{13}^{(2)}$	0.267	0.067	0.300	0.367
$\boldsymbol{X}_{13}^{(3)}$	0.201	0.300	0.133	0.367

From the Equation (10), we have the expected demand for various products under all possible demand states in the next period (i.e., period 13), that is

$$E(\boldsymbol{D}_{13}) = \begin{pmatrix} E(\xi_{1,13}) \\ E(\xi_{2,13}) \\ E(\xi_{3,13}) \end{pmatrix} = \begin{pmatrix} 0.033 & 0 & 0.633 & 0.333 \\ 0.267 & 0.067 & 0.300 & 0.367 \\ 0.201 & 0.300 & 0.133 & 0.367 \end{pmatrix} \begin{pmatrix} 100 \\ 300 \\ 500 \\ 700 \end{pmatrix} \approx \begin{pmatrix} 553 \\ 454 \\ 443 \end{pmatrix} \quad (24)$$

B. Sensitivity analysis.

In order to analyze the level of effects for the parameters on the optimal solution $(T_{n,13}^*, Q_{n,13}^*)$ for the product n ($n=1,2,3$), we perform the sensitivity analysis two important parameters m and β in the system. The results are shown in the following Tables 2 and 3.

TABLE 2. VALUES OF PARAMETER m EFFECTS ON $(T_{n,13}^*, Q_{n,13}^*)$ for product n ($n=1,2,3$)

	Product 1		Product 2		Product 3	
	$C_{1,13}=10$, $L_{13}=30$, $h_1=1.2$, $\beta_1=0.2$		$C_{2,13}=11$, $L_{13}=30$, $h_2=0.8$, $\beta_2=0.18$		$C_{3,13}=8$, $L_{13}=30$, $h_3=1.3$, $\beta_3=0.15$	
m	$T_{1,13}^*$	$Q_{1,13}^*$	$T_{2,13}^*$	$Q_{2,13}^*$	$T_{3,13}^*$	$Q_{3,13}^*$
2	0.70	144.57	0.93	172.22	0.91	104.50
3	0.83	264.54	1.02	290.32	1.00	202.07
4	0.90	401.00	1.09	412.52	1.07	320.30
5	0.95	539.75	1.13	526.34	1.13	445.82
6	1.00	671.39	1.17	625.48	1.17	567.85
7	1.03	790.86	1.20	707.98	1.20	679.47
8	1.06	896.14	1.22	774.34	1.23	777.10
9	1.08	987.04	1.23	826.23	1.24	859.61
10	1.09	1064.40	1.24	865.72	1.26	927.34

The computational results in the Table 2 that give some insight into the property for the optimal solution $(T_{n,13}^*, Q_{n,13}^*)$. As m varies and other parameters are constant, both of $T_{n,13}^*$ and $Q_{n,13}^*$ are increasing with the increase of m.

TABLE 3. VALUES OF PARAMETER β effects on $(T_{n,13}^*, Q_{n,13}^*)$ for product n ($n=1,2,$

	Product 1		Product 2		Product 3	
	$C_{1,13}=10$, $L_{13}=30$, $h_1=1.2$, $m=2$		$C_{2,13}=11$, $L_{13}=30$, $h_2=0.8$, $m=3$		$C_{3,13}=8$, $L_{13}=30$, $h_3=1.3$, $m=4$	
β	$T_{1,13}^*$	$Q_{1,13}^*$	$T_{2,13}^*$	$Q_{2,13}^*$	$T_{3,13}^*$	$Q_{3,13}^*$
0.1	0.94	245.00	1.19	353.12	1.15	191.7
0.2	0.73	144.57	0.96	217.93	0.92	146.93
0.3	0.71	93.97	0.95	131.63	0.88	94.48
0.4	0.72	65.70	0.98	85.25	0.89	63.04
0.5	0.76	48.58	1.02	59.15	0.90	44.62
0.6	0.72	31.76	1.06	43.45	1.10	33.30
0.7	0.84	30.01	1.10	33.41	0.95	25.96
0.8	0.88	24.69	1.14	26.64	0.97	20.96
0.9	0.92	20.78	1.18	21.88	0.98	17.41

As shown in the Table 3, we can conclude that $Q_{n,13}^*$ ($n=1,2,3$) is decreasing with the increase of β, that is, the result implies that $Q_{n,13}^*$ is a decreasing function with β. From the presentation of the computational results in the Table 3, we can't reveal the relationships between $T_{n,13}^*$ and β. However, by the formula $T_{nk} = Q^{\beta} / \beta D_{nk}$, we can derive a condition for the monotonic function between T_{nk} with β from calculating extreme points, that is, T_{nk} is increasing with the increase of β if $\beta > \ln e / \ln Q$.

V. CONCLUSIONS

Under the situation with competitiveness, the demands in the inventory system is related to each other tightly. Since substitutability is a common feature for most of products in the real-world competitive market, and diversion of the demand is a fundamental characteristic for the competitive products. Hence, improvement in prediction accuracy for multi-product attracts wide attention and becomes a challenging problem in both academia and practice.

In this paper, we take the multiple products stochastic demand series as the multivariate Markov processes, and propose a new demand forecasting method for multiple products with multivariate Markov chains, i.e., multivariate Markov demand model. The forecasting for demand in system are obtained by plugging the historical data (including demand state of each product) into the model. We can also obtain the relationships between the demands by the correlation coefficient matrix in the model. Then, we establish a multi-product EOQ model with the multivariate Markov demand for optimizing multi-product inventory system under the circumstances that the demands are dependent. Next, we perform an sensitivity analysis of the parameter β and m to test their effects to optimal solution $(T^*_{n,13}, Q^*_{n,13})$ .The computational results show that $T^*_{n,13}$ and $Q^*_{n,13}$ both are increasing in the nonlinearity factor *m*, but $Q^*_{n,13}$ is a decreasing function with the shape parameter β .

The results of the numerical example show that multivariate Markov demand model is an effective approach for optimizing multi-product inventory control system. Decision-maker in practice can use their historical demand data in their companies to make an optimal decision on the ordering quantity under the help of the multi-product EOQ model with multivariate Markov demand model.

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