The Similar Structure Method for Solving Boundary Value Problems of a Three Region Composite Bessel Equation

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***Abstract* — This paper studies the boundary value problem of three-region composite Bessel equation. Firstly, on the basis of similar structure of the solution of the boundary value problem of differential equations, the similar structure method for solving the class of composite boundary value problems is put forward and its steps are described. Secondly, the flow chart is contructed and the corresponding program is compiled. Finally, the new method is applied to solve a given boundary value problem of three-region composite Bessel equation. The curve of the solution of the boundary value problem is computed. The new method is simple and effective to solve this class of boundary value problems.**

Keywords - Boundary value problem; Three-region composite Bessel equation; Similar kernel function; Function of guide solution

I. INTRODUCTION

The boundary value problem of differential equation is widely applied in practical problems, especially in application of oil and gas well testing technology. The theory and interpretation method of well testing analysis has been developed rapidly. Therefore, the corresponding solving method of the boundary value problem of differential equation is also simultaneously required to be developed and improved. With the sustainable development and improvement of the theory of similar structure of solution of differential equations, its application in the field of engineering is also more widespread. So it plays an role in the solution to reservoir flow model.

At the beginning of this century, the thought of similar structure of the solution of the boundary value problem of differential equation began to form in Ref 5. Some gratifying results have been achieved. On the basis of the similar structure of solution of boundary value problem of differential equation, the researchers (Ref.1, Ref.4, Ref.6, Ref.14) simplified solutions of boundary value problems of second-order linear homogeneous differential equations and second-order partial differential equations. It can help to understand inherent the laws of analytic solution. Furthermore, the unification between the similarity of solution and the corresponding numbers and shapes has harmoniously improved the triad of mathematical theory.

With the analysis expression of fixed solution problems on a class of composite Bessel equation and composite modified Bessel equation, Ref.1 and Ref.14 gained the solution's formal similarity. This similar structure of solution explained that solutions of the class of equations can be given by the product of several fractions and the graphs have similarity too. By analyzing the solution of the reservoir pressure and dimensionless reservoir pressure distribution in Laplace space, which was aimed at the well test analytic model of composite reservoir, Ref.7 discovered the similar structure of the solution's form of the composite reservoir in the three kinds of outer boundary conditions (infinite, constant pressure and closed). Furthermore, this paper made the theoretical graph and analyzed the influence of the well-bore storage and skin effect on dimensionless reservoir pressure and dimensionless bottom-hole pressure by using numerical method of inversion.

Ref.15 studied the similar structure of solution in the Laplace space for a class of composite parabolic partial differential equations with the infinite outer boundary condition and the kind of inner boundary condition in connection with time . The work is essential to understand inherent laws of relevant engineering science and design practical analysis software.

Aiming at the boundary value problem of second-order homogeneous linear ordinary differential equations(systems) and the mixed problem for second order homogeneous linear partial differential equations(systems), Ref.8 reviewed some results of preliminary explorations with regard to the similar structure theory of their solutions or their Laplace space solutions and analyzed the formation idea of the similar solution and its significance of research and took apart the relation between the similar structure of its solution and governing equation and the boundary conditions. This paper also found a way to construct the solution of definite problems by making use of the formula of similar structure together with the similar kernel function and introduces the existing achievements which have been applied to permeation fluid mechanics.

The percolation model of composite reservoir was established in Ref.9, which considers the effective well-bore radius and well-bore storage. By Laplace transform, the exact solutions of reservoir pressure and bottom-hole pressure were obtained under different boundary conditions in Laplace space. According to the theory of similar structure of solution, Ref.9 defined the kernel functions which are only relevant to two linear independent solutions of governing equations and outer boundary conditions, what's more, different outer boundary conditions correspond to different similar kernel functions. There is a similar structure among the solutions under three outer boundaries.

Based on the analysis of the so1ution for a boundary value problem of the second-order linear homogeneous differential equation,Ref.10 studied the similar structure of solution and similar kernel functions, and put forward a new method of solving this class boundary value problem—the similar constructive method. The method is an innovative idea and a simple effective method of solving the boundary value problem of the differential equation.

The boundary value problem of two points was considered for second-order linear homogeneous differential equation in Ref.11. The existence and uniqueness theorems were proved. The similar structure and similar kernel function of the solutions were found.

Based on the analysis of the solution of a boundary value problem of second-order composite linear homogeneous differential equation, Ref.12 studied similar kernel functions and the similar structure of the solution and proposed a new method for solving this class boundary value problem.

Ref.3 solved a class of boundary value problems of the composite first Weber system. Solution with a form of continued fraction product to boundary value problem of the composite first Weber system was obtained. Then a new method was proposed solving the composite boundary value problem—Similar Constructing Method.

On the basis of similar structure of solution of a second-order linear differential equation' boundary value problem, Ref.2 proposed a new simple solution—similar constructive method of solution and summed up its detailed steps. A mathematical model of fractal reservoir with spherical flow under three kinds of outer boundary conditions (infinite, constant pressure and closed) was set up, in which influences of skin factor and well-bore storage are taken into consideration. And then SCMS is applied to solve it.

In this paper, the boundary value problem of three-region composite Bessel equation is studied. In section 1, the boundary value problem of three-region composite Bessel equation is given. In section 2, on the basis of similar structure of solution of boundary value problem of differential equation, a new method for solving the class of boundary value problem is proposed. In section 3, the steps of the method are summarized. In section 4, the new method is applied to solving a given boundary value problem of three-region composite Bessel equation and the curve of the solution of the boundary value problem is drawn.

II. PROPOSED BOUNDARY VALUE PROBLEMS AND PRELIMINARY KNOWLEDGE

In this paper, the following boundary value problem of three-region composite Bessel equation is studied:

$$\begin{cases} x^2 y_1'' + x y_1' + \left(x^2 - \nu_1^2\right) y_1 = 0, & \left(a \le x \le b\right) \\ x^2 y_2'' + x y_2' + \left(x^2 - \nu_2^2\right) y_2 = 0 & \left(b \le x \le c\right) \\ x^2 y_3'' + x y_3' + \left(x^2 - \nu_3^2\right) y_3 = 0 & \left(c \le x \le d\right) \\ \left[E y_1 + \left(1 + EF\right) y_1'\right]_{x=a} = D \\ y_1\big|_{x=b} = \lambda_1 \, y_2\big|_{x=b} , y_1'\big|_{x=b} = \lambda_2 \, y_2'\big|_{x=b} \\ y_2\big|_{x=c} = \mu_1 \, y_3\big|_{x=c} , y_2'\big|_{x=c} = \mu_2 \, y_3'\big|_{x=c} \\ \left[M y_3 + N y_3'\right]_{x=d} = 0 \end{cases} \tag{1.1}$$

Where D , E , F , M , N , a , b , c , d , λ_1 , λ_2 , μ_1 , μ_2 , ν_1 , ν_2 , ν_3 are constants and $\lambda_1\lambda_2\mu_1\mu_2 \ne 0$, $M^2 + N^2 \ne 0$ and $D \ne 0$.

$J_{\nu_i}\left(x\right)$ and $Y_{\nu_i}\left(x\right)$ $\left(i = 1,2,3\right)$ are two linear independent solutions of second-order linear differential equations $x^2 y_i'' + x y_i' + \left(x^2 - \nu_i^2\right) y_i = 0 \left(x > 0, i = 1,2,3\right)$. $J_n\left(\cdot\right)$, $Y_n\left(\cdot\right)$ are respectively the first and the second class of Bessel functions of order n [15]. Defining a binary function as below:

$$\varphi_{m,n}\left(x,\xi\right) = J_m\left(x\right) Y_n\left(\xi\right) + (-1)^{m-n+1} Y_m\left(x\right) J_n\left(\xi\right) \tag{1.2}$$

This paper leads into functions of guide solution as follow:

$$\varphi_{0,0}^i\left(x,\xi\right) @ \varphi_{\nu_i,\nu_i}\left(x,\xi\right) \quad \left(i = 1,2,3\right) \tag{1.3}$$

$$\begin{aligned} \varphi_{1,0}^i\left(x,\xi\right) & @ \frac{\partial}{\partial x} \varphi_{\nu_i,\nu_i}\left(x,\xi\right) \\ & = \frac{\nu_i}{x} \varphi_{\nu_i,\nu_i}\left(x,\xi\right) - \varphi_{\nu_i+1,\nu_i}\left(x,\xi\right) \quad \left(i = 1,2,3\right) \end{aligned} \tag{1.4}$$

$$\begin{aligned} \varphi_{0,1}^i\left(x,\xi\right) & @ \frac{\partial}{\partial \xi} \varphi_{\nu_i,\nu_i}\left(x,\xi\right) \\ & = \frac{\nu_i}{\xi} \varphi_{\nu_i,\nu_i}\left(x,\xi\right) - \varphi_{\nu_i,\nu_i+1}\left(x,\xi\right) \quad \left(i = 1,2,3\right) \end{aligned} \tag{1.5}$$

$$\begin{aligned} \varphi_{1,1}^i\left(x,\xi\right) & @ \frac{\partial^2}{\partial x \partial \xi} \varphi_{\nu_i,\nu_i}\left(x,\xi\right) \\ & = \frac{\nu_i^2}{x\xi} \varphi_{\nu_i,\nu_i}\left(x,\xi\right) - \frac{\nu_i}{\xi} \varphi_{\nu_i+1,\nu_i}\left(x,\xi\right) - \frac{\nu_i}{x} \varphi_{\nu_i,\nu_i+1}\left(x,\xi\right) + \varphi_{\nu_i+1,\nu_i+1}\left(x,\xi\right) \end{aligned} \tag{1.6}$$

$\left(i = 1,2,3\right)$

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Where $i=1$ denotes inner region $(a \le x \le b)$, $i=2$ denotes middle region $(b \le x \le c)$, $i=3$ denotes outer region $(c \le x \le d)$.

III. THE MAIN THEOREM AND ITS PROOF

Theorem If the boundary value problem (1) has unique solution, then solutions of inner, middle and outer regions are expressed respectively as follows (the detailed proof process is presented in Appendix A):

$$y_1(x) = D \cdot \frac{1}{E + \dfrac{1}{F + \Phi_1(a)}} \cdot \frac{1}{F + \Phi_1(a)} \cdot \Phi_1(x) \quad (a \le x \le b) \tag{2.1}$$

$$y_2(x) = D \cdot \frac{1}{E + \dfrac{1}{F + \Phi_1(a)}} \cdot \frac{1}{F + \Phi_1(a)} \cdot \frac{\varphi_{0,1}^1(b,b)}{\lambda_1 \Phi_2(b) \varphi_{1,1}^1(a,b) - \lambda_2 \varphi_{1,0}^1(a,b)} \cdot \Phi_2(x) \quad (b \le x \le c) \tag{2.2}$$

$$y_3(x) = D \cdot \frac{1}{E + \dfrac{1}{F + \Phi_1(a)}} \cdot \frac{1}{F + \Phi_1(a)} \times \frac{\varphi_{0,1}^1(b,b)\varphi_{0,1}^2(c,c)}{\left[\lambda_1 \Phi_2(b) \varphi_{1,1}^1(a,b) - \lambda_2 \varphi_{1,0}^1(a,b)\right]\left[\mu_1 \Phi_3(c) \varphi_{1,1}^2(b,c) - \mu_2 \varphi_{1,0}^2(b,c)\right]} \cdot \Phi_3(x) \quad (c \le x \le d) \tag{2.3}$$

Where $\Phi_3(x)$, $\Phi_2(x)$ and $\Phi_1(x)$ are called similar kernel functions of outer, middle and inner regions respectively and they are expressed as follows:

$$\Phi_3(x) = \frac{M \varphi_{0,0}^3(x,d) + N \varphi_{0,1}^3(x,d)}{M \varphi_{1,0}^3(c,d) + N \varphi_{1,1}^3(c,d)} \quad (c \le x \le d) \tag{2.4}$$

$$\Phi_2(x) = \frac{\mu_2 \varphi_{0,0}^2(x,c) - \mu_1 \Phi_3(c) \varphi_{0,1}^2(x,c)}{\mu_2 \varphi_{1,0}^2(b,c) - \mu_1 \Phi_3(c) \varphi_{1,1}^2(b,c)} \quad (b \le x \le c) \tag{2.5}$$

$$\Phi_1(x) = \frac{\lambda_2 \varphi_{0,0}^1(x,b) - \lambda_1 \Phi_2(b) \varphi_{0,1}^1(x,b)}{\lambda_2 \varphi_{1,0}^1(a,b) - \lambda_1 \Phi_2(b) \varphi_{1,1}^1(a,b)} \quad (a \le x \le b) \tag{2.6}$$

Corollary 1 In the boundary value problem (1.1), if the outer boundary condition is $y_3(d)=0$ (i.e. $M \ne 0, N = 0$), the corresponding similar kernel function of outer region is

$$\Phi_3(x) = \varphi_{0,0}^3(x,d) \big/ \varphi_{1,0}^3(c,d).$$

Corollary 2 In the boundary value problem (1.1), if the outer boundary condition is $y_3'(d)=0$ (i.e. $M = 0, N \ne 0$),

the corresponding similar kernel function of outer region is

$$\Phi_3(x) = \varphi_{0,1}^3(x,d) \big/ \varphi_{1,1}^3(c,d).$$

Corollary 3 In the boundary value problem (1.1), if the left boundary condition is the second boundary condition (i.e. $y_1'\big|_{x=a} = 1$), the solution of inner region of the boundary value problem (1.1) is the similar kernel function of inner region (i.e. $\Phi_1(x)$).

Corollary 4 The first continued fraction, which belongs to the structure of the solution (i.e. Eq. (2.1)) to the boundary value problem (1.1), has the following property (The detailed process of proving in Appendix A):

$$\left[y_1(x) + F y_1'(x)\right]_{x=a} = \frac{D}{E + \dfrac{1}{F + \Phi_1(a)}}$$

IV. STEPS OF THE NEW METHOD

According to solution procedures of the above boundary value problem, it is easy to induce the steps of the new method for solving the boundary value problem of three-region composite Bessel equation. Detailed steps are as follows:

Step1. Solving governing equations

Two linear independent solutions $J_{v_i}(x), Y_{v_i}(x)$ $(i=1,2,3)$ of governing equations are obtained by solving governing equations of inner, middle and outer regions of the boundary value problem (1.1) respectively.

Step2. Constructing functions of guide solution

We construct the functions of guide solution of inner, middle and outer regions $\varphi_{0,0}^i(x,\xi)(i=1,2,3)$ by using two linear independent solutions $J_{v_i}(x), \mathrm{Y}_{v_i}(x)$ $(i=1,2,3)$ of governing equations of inner, middle and outer regions of the boundary value problem (1.1) respectively, as shown Eq.(1.3). Other functions of guide solution can be obtained by calculating partial derivatives of $\varphi_{0,0}^i(x,\xi)(i=1,2,3)$ to x, ξ respectively, as shown Eq.(1.4)-(1.6).

Step3. Constructing similar kernel functions of inner, middle and outer regions

Firstly, the similar kernel function $\Phi_3(x)$ of outer region of the boundary value problem (1.1) can be structured by using functions of guide solution of outer region and coefficients M , N of the homogeneous outer boundary condition, as shown Eq.(2.4). After that, the value of $\Phi_3(c)$ is calculated. Secondly, the similar kernel function $\Phi_2(x)$ of middle region can be structured by using functions of guide solution of middle region, coefficients μ_1 , μ_2 of two convergence conditions of middle and outer region and $\Phi_3(c)$, as shown Eq.(2.5). After that, the value of $\Phi_2(b)$ is calculated. Finally, the similar kernel function $\Phi_1(x)$ of inner region of the boundary value problem (1.1) can be structured by using functions of guide solution of inner region, coefficients λ_1 , λ_2 of two convergence conditions of inner and middle regions and $\Phi_2(c)$, as shown Eq.(2.6). After that, the value of $\Phi_1(a)$ is calculated.

Step4. Obtaining solution of the boundary value problem

According to Eq.(2.1), Eq.(2.2) and Eq.(2.3), solutions of the inner, middle and outer regions are obtained by assembling coefficients D , E , F of the non-homogeneous inner boundary condition, similar kernel functions $\Phi_1(x)$, $\Phi_2(x)$, $\Phi_3(x)$, values of $\Phi_1(a)$, $\Phi_2(b)$, $\Phi_3(c)$, functions of guide solution of inner and middle regions, coefficients λ_1, λ_2, μ_1, μ_2 of four convergence conditions of three regions , respectively.

The flow chart (Fig. 1) of algorithm of steps of the above method is:

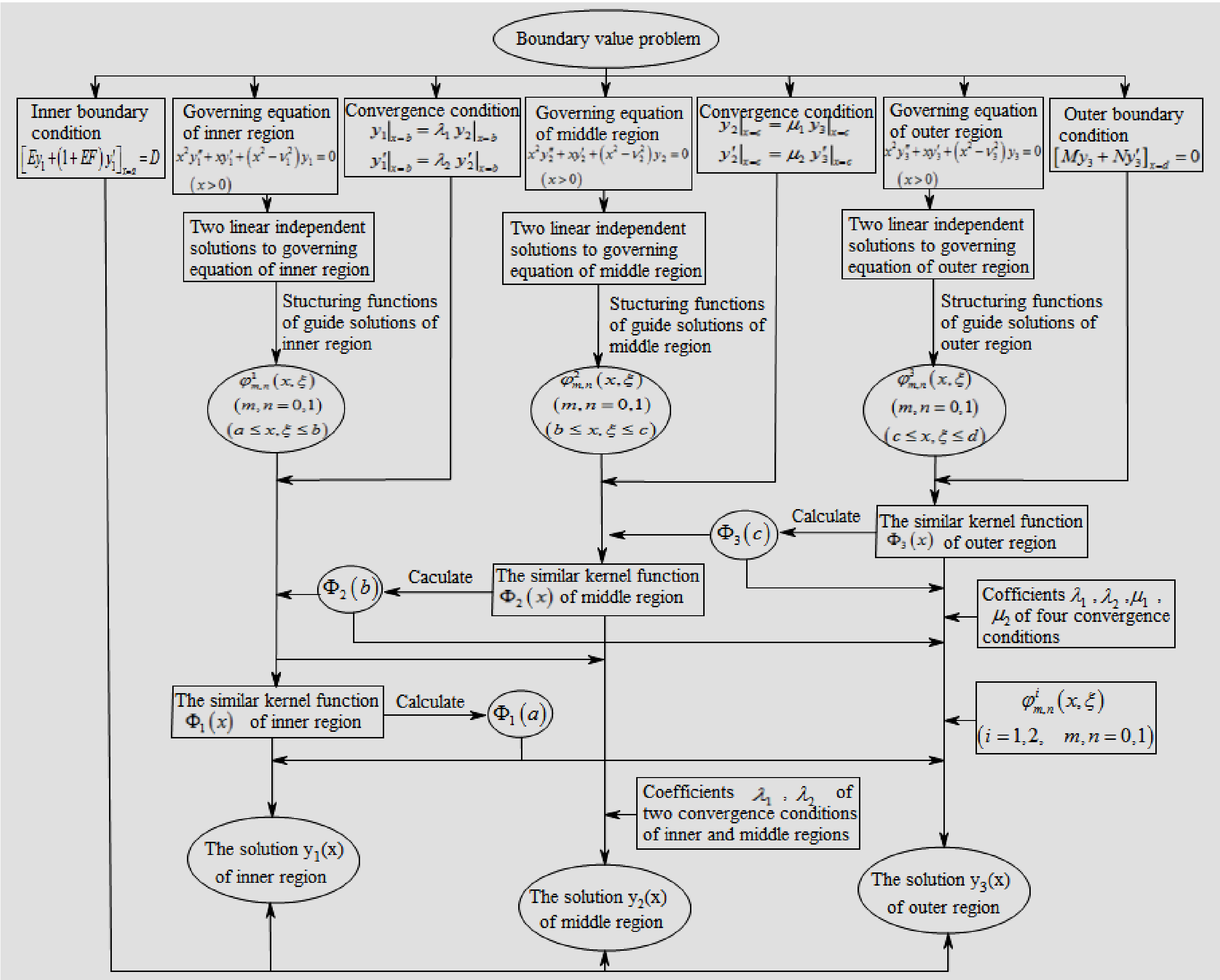

Fig.1 The flow chart of algorithm of the method.

The flow chart of algorithm clearly presents the relationship between solutions of three regions and similar kernel functions, functions of guide solution, boundary conditions and connection conditions. It elaborately illustrates the solution procedure of solving the class of boundary value problems of three-region composite Bessel equation.

V. THE EXAMPLE

The following boundary value problem is solved:

$$\begin{cases} x^2y_1''+xy_1'+x^2y_1=0, & (1\le x\le 4) \\ x^2y_2''+xy_2'+\left(x^2-1\right)y_2=0, & (4\le x\le 8) \\ x^2y_3''+xy_3'+\left(x^2-4\right)y_3=0, & (8\le x\le 12) \\ 2y_1\big|_{x=1}+7y_1'\big|_{x=1}=2 \\ y_1\big|_{x=4}=y_2\big|_{x=4},\ y_1'\big|_{x=4}=2y_2'\big|_{x=4} \\ y_2\big|_{x=8}=y_3\big|_{x=8},\ y_2'\big|_{x=8}=2y_3'\big|_{x=8} \\ 3y_3\big|_{x=12}+2y_3'\big|_{x=12}=0 \end{cases} \tag{4.1}$$

Comparing with the boundary value problem (1.1) and (4.1), we know that $\nu_1=0$, $\nu_2=1$, $\nu_3=2$, $a=1$, $b=4$, $c=8$, $d=12$, $\lambda_1=1$, $\lambda_2=2$, $\mu_1=1$, $\mu_2=2$, $D=2$, $E=2$, $F=3$, $M=3$, $N=2$. The boundary value problem (4.1) has unique solution (the result is given in Appendix B). According to the new method, we solve the boundary value problem (4.1).

Step1. Solving governing equations

Two linear independent solutions $J_{\nu_i}(x)$, $Y_{\nu_i}(x)$ $(i=1,2,3)$ [15] of governing equations are obtained by solving governing equations of inner, middle and outer regions of the boundary value problem (4.1) , respectively .

Step2. Constructing functions of guide solution

According to Eqs.(1.2)-(1.6), functions of guide solution of inner, middle and outer regions are structured by using two linear independent solutions $J_{\nu_i}(x)$, $Y_{\nu_i}(x)$ $(i=1,2,3)$ of governing equations of inner, middle and outer regions of the boundary value problem (4.1) respectively as follows:

$$\varphi_{0,0}^1(x,\xi)=J_0(x)Y_0(\xi)-Y_0(x)J_0(\xi)$$

$$\varphi_{0,1}^1(x,\xi)=-J_0(x)Y_1(\xi)+Y_0(x)J_1(\xi)$$

$$\varphi_{1,0}^1(x,\xi)=-J_1(x)Y_0(\xi)+Y_1(x)J_0(\xi)$$

$$\varphi_{1,1}^1(x,\xi)=J_1(x)Y_1(\xi)-Y_1(x)J_1(\xi)$$

$$\varphi_{0,0}^2(x,\xi)=J_1(x)Y_1(\xi)-Y_1(x)J_1(\xi)$$

$$\varphi_{0,1}^2(x,\xi)=\frac{1}{\xi}\left[J_1(x)Y_1(\xi)-Y_1(x)J_1(\xi)\right]-\left[J_1(x)Y_2(\xi)-Y_1(x)J_2(\xi)\right]$$

$$\varphi_{1,0}^2(x,\xi)=\frac{1}{x}\left[J_1(x)Y_1(\xi)-Y_1(x)J_1(\xi)\right]-\left[J_2(x)Y_1(\xi)-Y_2(x)J_1(\xi)\right]$$

$$\varphi_{1,1}^2(x,\xi)=\frac{1}{x\xi}\left[J_1(x)Y_1(\xi)-Y_1(x)J_1(\xi)\right]-\frac{1}{\xi}\left[J_2(x)Y_1(\xi)-Y_2(x)J_1(\xi)\right]$$
$$-\frac{1}{x}\left[J_1(x)Y_2(\xi)-Y_1(x)J_2(\xi)\right]+\left[J_2(x)Y_2(\xi)-Y_2(x)J_2(\xi)\right]$$

$$\varphi_{0,0}^3(x,\xi)=J_2(x)Y_2(\xi)-Y_2(x)J_2(\xi)$$

$$\varphi_{0,1}^3(x,\xi)=\frac{2}{\xi}\left[J_2(x)Y_2(\xi)-Y_2(x)J_2(\xi)\right]-\left[J_2(x)Y_3(\xi)-Y_2(x)J_3(\xi)\right]$$

$$\varphi_{1,0}^3(x,\xi)=\frac{2}{x}\left[J_2(x)Y_2(\xi)-Y_2(x)J_2(\xi)\right]-\left[J_3(x)Y_2(\xi)-Y_3(x)J_2(\xi)\right]$$

$$\varphi_{1,1}^3(x,\xi)=\frac{4}{x\xi}\left[J_2(x)Y_2(\xi)-Y_2(x)J_2(\xi)\right]-\frac{2}{\xi}\left[J_3(x)Y_2(\xi)-Y_3(x)J_2(\xi)\right]$$
$$-\frac{1}{x}\left[J_2(x)Y_3(\xi)-Y_2(x)J_3(\xi)\right]+\left[J_3(x)Y_3(\xi)-Y_3(x)J_3(\xi)\right]$$

Step3. Constructing similar kernel functions of inner, middle and outer regions

According to the Eq.(2.4), the similar kernel function of outer region of the boundary value problem (4.1) is structured as follows:

$$\Phi_3(x)=\frac{3\varphi_{0,0}^3(x,12)+2\varphi_{0,1}^3(x,12)}{3\varphi_{1,0}^3(8,12)+2\varphi_{1,1}^3(8,12)}\quad(8\le x\le 12)$$

Thus

$$\Phi_3(8)=\frac{3\varphi_{0,0}^3(8,12)+2\varphi_{0,1}^3(8,12)}{3\varphi_{1,0}^3(8,12)+2\varphi_{1,1}^3(8,12)}$$

According to the Eq.(2.5), the similar kernel function of middle region of the boundary value problem (4.1) is structured as follows:

$$\Phi_2(x)=\frac{2\varphi_{0,0}^2(x,8)-\Phi_3(8)\varphi_{0,1}^2(x,8)}{2\varphi_{1,0}^2(4,8)-\Phi_3(8)\varphi_{1,1}^2(4,8)}\quad(4\le x\le 8).$$

Thus

$$\Phi_2(4)=\frac{2\varphi_{0,0}^2(4,8)-\Phi_3(8)\varphi_{0,1}^2(4,8)}{2\varphi_{1,0}^2(4,8)-\Phi_3(8)\varphi_{1,1}^2(4,8)}.$$

According to the Eq.(2.6), the similar kernel function of inner region of the boundary value problem (4.1) is structured as follows:

$$\Phi_1(x)=\frac{2\varphi_{0,0}^1(x,4)-\Phi_2(4)\varphi_{0,1}^1(x,4)}{2\varphi_{1,0}^1(1,4)-\Phi_2(4)\varphi_{1,1}^1(1,4)}\quad(1\le x\le 4).$$

Thus

$$\Phi_1(1)=\frac{2\varphi_{0,0}^1(1,4)-\Phi_2(4)\varphi_{0,1}^1(1,4)}{2\varphi_{1,0}^1(1,4)-\Phi_2(4)\varphi_{1,1}^1(1,4)}.$$

Step4. Obtaining the solution of the boundary value problem (4.1)

According to Eq.(2.1), Eq.(2.2) and Eq.(2.3), solutions of inner, middle and outer regions of the boundary value problem (4.1) can be obtained respectively as follows:

$$y_1(x)=\frac{2}{7+2\cdot\Phi_1(1)}\cdot\Phi_1(x)\quad(1\le x\le 4)$$

$$y_2(x)=\frac{2}{7+2\cdot\Phi_1(1)}\cdot\frac{\varphi_{0,1}^1(4,4)}{\Phi_2(4)\varphi_{1,1}^1(1,4)-2\varphi_{1,0}^1(1,4)}\cdot\Phi_2(x)\quad(4\le x\le 8)$$

$$y_3(x)=\frac{2}{7+2\cdot\Phi_1(1)}\cdot\frac{\varphi_{0,1}^1(4,4)\varphi_{0,1}^2(8,8)}{\left[\Phi_2(4)\varphi_{1,1}^1(1,4)-2\varphi_{1,0}^1(1,4)\right]\left[\Phi_3(8)\varphi_{1,1}^2(4,8)-2\varphi_{1,0}^2(4,8)\right]}\cdot\Phi_3(x)$$
$$(8\le x\le 12)$$

According to the flow chart of the algorithm of the method, corresponding program is compiled by using the MATLAB language. Then the curve of the solution (Fig.2) of the boundary problem (4.1) is drawn by running the program on the computer as below:

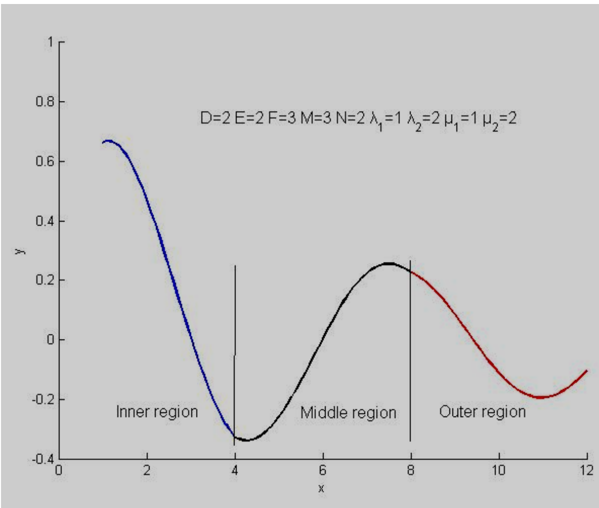

Fig.2 The curve of solution of the boundary value problem (4.1)

VI. CONCLUSIONS

(1) When dealing with the boundary value problem of three-region composite Bessel equation, only two linear independent solutions $J_{\nu_i}(x), Y_{\nu_i}(x)\ (i=1,2,3)$ of governing equations of inner, middle and outer regions respectively are obtained. Then the boundary value problem can be solved by the obtained method.

(2) It is clear that the method is a convenient, effective and creative way to solve the boundary value problem of three-region composite Bessel equation.

(3) Similar structures of solution of inner, middle and outer regions clearly show the relationship between solutions and similar kernel functions, functions of guide solution that is generated by using two linear independent solutions of governing equations, boundary conditions and connection conditions.

(4) According the new method, a corresponding program is compiled. It is applied to drawing a graph of the solution of the boundary value problem (4.1). And the graph clearly illustrates the solution of the boundary value problem.

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REFERENCES

[1] CHEN Zi-chun, LIU Peng-hui, LI Shun-chu. The Similar Structure of Composite Bessel Equation on Fixed Solution Problem (in Chinese) [J]. Journal of Chongqing Techno Business University (Natural Science Edition), 2006, 23(1):1-4(to 24).

[2] Cui-Cui Sheng, Jin-Zhou Zhao, Yong-Ming Li, Shun-Chu Li and Hu Jia. Similar Construction Method of Solution for Solving the Mathematical Model of Fractal Reservoir with Spherical Flow [J]. Journal of Applied Mathematics, vol. 2013, Article ID 219218, 8 pages, 2013.

[3] Dong Xiaoxu, Li Shunchu, Gui Dongdong, Pu Jun, Li Huichun. Similar Constructing Method for Solving the Boundary Value Problem of the Composite First Weber equation [J]. American Journal of Applied Mathematics and Statistics, 2013, 1(4):76-82.

[4] JIA Minhui, LI Shunchu. The Similar Structure of Solution Differential Equation on Boundary Value Problem (in Chinese) [J]. College Mathematics, 2005, 21(5):37-39.

[5] Li Shunchu, Jia Minhui. The Formal Similarity of Solutions on the Class of Differential Equation [J]. Journal of UEST of China, 2004, 33(Supp.):95-98.

[6] LI Shun-chu. The Formal Similarity of Solutions to the Class of 2-order Partial Differential Equation in the Laplace Space (in English) [J]. Journal of Xihua University (Natural Science Edition), 2007, 26(4):83-86.

[7] LI Shun-chu, ZHENG Peng-she, ZHANG Yu-fei. The Similar Structure of Pressure Distribution in the Composite Reservoir (in Chinese) [J]. Journal of Mathematics in Practice and Theory, 2008, 38(3):23-28.

[8] LI Shun-chu. Preliminary Exploration and Prospects of the Similar Structure of Solutions of Differential Equations (in Chinese) [J]. Journal of Xihua University (Natural Science Edition), 2010, 29(2):223-226(to238).

[9] Li Quanyong, Li Shunchu, Li Wei, Tang Yibin. Study on percolation model of composite media reservoir based on similar structure of solution (in Chinese) [J]. Fault-Block Oil & Gas Field, 2011, 18(5): 623-625.

[10] Li Shunchu, Liao Zhijian. Constructing the Solution of Boundary Value Problem of the Differential Equation with its an Arbitrary Non-trivial Solution (in Chinese) [J]. Journal of Sichuan University (Natural Science Edition), 2012, 49(6):1209-1213.

[11] Li Shun-chu, Wu Xiao-qing. Several Important Properties for the Boundary Value Problems of Second-order Linear Homogeneous Differential Equation (in Chinese)[J]. Journal of Xihua University (Natural Science Edition), 2013, 32(1): 23-26.

[12] Li Shunchu. The Similarity Structuring Method of Boundary Value Problems for Composite Differential Equations [J]. Journal of Xihua University (Natural science Edition), 2013, 32(1):23-26.

[13] Liu Shishi, Liu Shida. Special Function [M]. Beijing: China Meteorological Press, 2002.

[14] LIU Peng-hui, CHEN Zi-chun, LI Shun-chu. Similar Structure of Fixed Solution Problems for Composite Abnormal Bessel Equations (in Chinese) [J]. Journal of Xihua University (Natural Science Edition), 2006, 25(2):23-26.

[15] SU Jian-peng, LI Shun-chu, LI Cheng-jie. The Similar of Solutions in the Laplace Space of Composite Parabolic Partial Differential Equation [J]. Journal of Zaozhuang University, 2009, 26(2):6-11.

APPENDIX A

In this section, the detailed proof of theorem is discussed.

$J_{\nu_i}(x)$ and $Y_{\nu_i}(x)$ are two linear independent solutions of second-order linear homogeneous differential equations $x^2 y_i'' + x y_i' + \left(x^2 - \nu_i^2\right) y_i = 0, \quad (x>0) \quad (i=1,2,3)$. It is universally known that general solutions of governing equations of inner, middle and outer regions of the boundary value problem (1.1) are [15]

$$y_i(x) = A_i J_{\nu_i}(x) + B_i Y_{\nu_i}(x) \quad (i=1,2,3) \tag{A.1}$$

By substituting Eq.(A.1) into inner, middle and outer boundary conditions and four connection conditions of the boundary value problem (1.1), we obtain the following equations respectively:

$$\left\{E J_{\nu_1}(a) + (1+EF)\left[\frac{\nu_1}{a} J_{\nu_1}(a) - J_{\nu_1+1}(a)\right]\right\} A_1 + \left\{E Y_{\nu_1}(a) + (1+EF)\left[\frac{\nu_1}{a} Y_{\nu_1}(a) - Y_{\nu_1+1}(a)\right]\right\} B_1 = D$$

(A.2)

$$J_{\nu_1}(b) A_1 + Y_{\nu_1}(b) B_1 - \lambda_1 J_{\nu_2}(b) A_2 - \lambda_1 Y_{\nu_2}(b) B_2 = 0$$

(A.3)

$$\left[\frac{\nu_1}{b} J_{\nu_1}(b) - J_{\nu_1+1}(b)\right] A_1 + \left[\frac{\nu_1}{b} Y_{\nu_1}(b) - Y_{\nu_1+1}(b)\right] B_1 - \lambda_2\left[\frac{\nu_2}{b} J_{\nu_2}(b) - J_{\nu_2+1}(b)\right] A_2 - \lambda_2\left[\frac{\nu_2}{b} Y_{\nu_2}(b) - Y_{\nu_2+1}(b)\right] B_2 = 0$$

(A.4)

$$J_{\nu_2}(c) A_2 + Y_{\nu_2}(c) B_2 - \mu_1 J_{\nu_3}(c) A_3 - \mu_1 Y_{\nu_3}(c) B_3 = 0$$

(A.5)

$$\left[\frac{\nu_2}{c} J_{\nu_2}(c) - J_{\nu_2+1}(c)\right] A_2 + \left[\frac{\nu_2}{c} Y_{\nu_2}(c) - Y_{\nu_2+1}(c)\right] B_2 - \mu_2\left[\frac{\nu_3}{c} J_{\nu_3}(c) - J_{\nu_3+1}(c)\right] A_3 - \mu_2\left[\frac{\nu_3}{c} Y_{\nu_3}(c) - Y_{\nu_3+1}(c)\right] B_3 = 0$$

(A.6)

$$\left\{M J_{\nu_3}(d) + N\left[\frac{\nu_3}{d} J_{\nu_3}(d) - J_{\nu_3+1}(d)\right]\right\} A_3 + \left\{M Y_{\nu_3}(d) + N\left[\frac{\nu_3}{d} Y_{\nu_3}(d) - Y_{\nu_3+1}(d)\right]\right\} B_3 = 0$$

(A.7)

According to the existence and uniqueness of solution of the boundary value problem (1.1), we know that the coefficient determinant Δ of linear equations (Eqs.(A.2)-(A.7)) about undetermined coefficients is not equal to zero, and

$$\begin{aligned}\Delta = E\Big[&\lambda_2\mu_2 M\varphi_{0,0}^1(a,b)\varphi_{1,0}^2(b,c)\varphi_{1,0}^3(c,d) - M\lambda_2\mu_1\varphi_{0,0}^1(a,b)\varphi_{1,1}^2(b,c)\varphi_{0,0}^3(c,d)\\ &- \lambda_1\mu_2 M\varphi_{0,1}^1(a,b)\varphi_{0,0}^2(b,c)\varphi_{1,0}^3(c,d) + \lambda_1\mu_1 M\varphi_{0,1}^1(a,b)\varphi_{0,1}^2(b,c)\varphi_{0,0}^3(c,d)\\ &+ \lambda_2\mu_2 N\varphi_{0,0}^1(a,b)\varphi_{1,0}^2(b,c)\varphi_{1,1}^3(c,d) - \lambda_2\mu_1 N\varphi_{0,0}^1(a,b)\varphi_{1,1}^2(b,c)\varphi_{0,1}^3(c,d)\\ &- \lambda_1\mu_2 N\varphi_{0,1}^1(a,b)\varphi_{0,0}^2(b,c)\varphi_{1,1}^3(c,d) + \lambda_1\mu_1 N\varphi_{0,1}^1(a,b)\varphi_{0,1}^2(b,c)\varphi_{0,1}^3(c,d)\Big]\\ &+ (1+EF)\big[\lambda_2\mu_2 M\varphi_{1,0}^1(a,b)\varphi_{1,0}^2(b,c)\varphi_{1,0}^3(c,d) - \lambda_2\mu_1 M\varphi_{1,0}^1(a,b)\varphi_{1,1}^2(b,c)\\ &\cdot\varphi_{0,0}^3(c,d) - \lambda_1\mu_2 M\varphi_{1,1}^1(a,b)\varphi_{0,0}^2(b,c)\varphi_{1,0}^3(c,d) + \lambda_1\mu_1 M\varphi_{1,1}^1(a,b)\varphi_{0,1}^2(b,c)\\ &\varphi_{0,0}^3(c,d) + \lambda_2\mu_2 N\varphi_{1,0}^1(a,b)\varphi_{1,0}^2(b,c)\varphi_{1,1}^3(c,d) - \lambda_2\mu_1 N\varphi_{1,0}^1(a,b)\varphi_{1,1}^2(b,c)\\ &\cdot\varphi_{0,1}^3(c,d) - \lambda_1\mu_2 N\varphi_{1,1}^1(a,b)\varphi_{0,0}^2(b,c)\varphi_{1,1}^3(c,d) + \lambda_1\mu_1 N\varphi_{1,1}^1(a,b)\varphi_{0,1}^2(b,c)\\ &\varphi_{0,1}^3(c,d)\big]\end{aligned}$$

(A.8)

Values of $A_1, B_1, A_2, B_2, A_3, B_3$ can be obtained by using the Cramer rule as follows:

$$A_1 = D\left\{Y_{\nu_1}(b) - \left[\frac{\nu_1}{b} Y_{\nu_1}(b) - Y_{\nu_1+1}(b)\right]\Phi_2(b)\right\} \times \Big[E\varphi_{0,0}^1(a,b) - E\varphi_{0,1}^1(a,b)\Phi_2(b) + (1+EF)\varphi_{1,0}^1(a,b) - (1+EF)\varphi_{1,1}^1(a,b)\Phi_2(b)\Big]^{-1}$$

(A.9)

$$B_1 = -D\left\{J_{\nu_1}(b) - \left[\frac{\nu_1}{b} J_{\nu_1}(b) - J_{\nu_1+1}(b)\right]\Phi_2(b)\right\} \times \Big[E\varphi_{0,0}^1(a,b) - E\varphi_{0,1}^1(a,b)\Phi_2(b) + (1+EF)\varphi_{1,0}^1(a,b) - (1+EF)\varphi_{1,1}^1(a,b)\Phi_2(b)\Big]^{-1}$$

(A.10)

$$\begin{aligned}A_2 = \frac{D}{\Delta}\Big\{&-M\mu_2 Y_{\nu_2}(c)\varphi_{0,1}^1(b,b)\varphi_{1,0}^3(c,d) + M\mu_1\left[\frac{\nu_2}{c} Y_{\nu_2}(c) - Y_{\nu_2+1}(c)\right]\varphi_{0,1}^1(b,b)\\ &\cdot\varphi_{0,0}^3(c,d) - N\mu_2 Y_{\nu_2}(c)\varphi_{0,1}^1(b,b)\varphi_{1,1}^3(c,d) + N\mu_1\left[\frac{\nu_2}{c} Y_{\nu_2}(c) - Y_{\nu_2+1}(c)\right]\\ &\cdot\varphi_{0,1}^1(b,b)\varphi_{0,1}^3(c,d)\end{aligned}$$

(A.11)

$$B_2=-\frac{D}{\Delta}\left\{-M\mu_2 J_{\nu_2}(c)\varphi_{0,1}^1(b,b)\varphi_{1,0}^3(c,d)+M\mu_1\left[\frac{\nu_2}{c}J_{\nu_2}(c)-J_{\nu_2+1}(c)\right]\right.$$
$$\cdot\varphi_{0,1}^1(b,b)\varphi_{0,0}^3(c,d)-N\mu_2 J_{\nu_2}(c)\varphi_{0,1}^1(b,b)\varphi_{1,1}^3(c,d)+N\mu_1$$
$$\left.\cdot\left[\frac{\nu_2}{c}J_{\nu_2}(c)-J_{\nu_2+1}(c)\right]\varphi_{0,1}^1(b,b)\varphi_{0,1}^3(c,d)\right\} \quad (A.12)$$

$$A_3=\frac{D}{\Delta}\left\{MY_{\nu_3}(d)+N\left[\frac{\nu_3}{d}Y_{\nu_3}(d)-Y_{\nu_3+1}(d)\right]\right\}\varphi_{0,1}^1(b,b)\varphi_{0,1}^2(c,c) \quad (A.13)$$

$$B_3=-\frac{D}{\Delta}\left\{MJ_{\nu_3}(d)+N\left[\frac{\nu_3}{d}J_{\nu_3}(d)-J_{\nu_3+1}(d)\right]\right\}\varphi_{0,1}^1(b,b)\varphi_{0,1}^2(c,c) \quad (A.14)$$

By substituting values of A_1,B_1,A_2,B_2,A_3,B_3 (Eqs.(A.9)-(A.14)) into Eq.(A.1) and using the similar kernel function of outer region Eq.(2.4), the similar kernel function of middle region Eq.(2.5) and the similar kernel function of inner region Eq.(2.6), solutions of inner, middle and outer regions of the boundary value problem (1.1) are obtained respectively, i.e. Eq.(2.1), Eq.(2.2) and Eq.(2.3).

APPENDIX B

In this section, the boundary value problem (4.1) has unique solution, which is proved.

$J_0(x)$ and $Y_0(x)$ are two linear independent solutions governing equation of inner region $x^2y_1''+xy_1'+(x^2)y_1=0$. $J_1(x)$ and $Y_1(x)$ are two linear independent solutions governing equation of middle region $x^2y_2''+xy_2'+(x^2-1)y_2=0$. $J_2(x)$ and $Y_2(x)$ are two linear independent solutions governing equation of outer region $x^2y_3''+xy_3'+(x^2-4)y_3=0$. According to Appendix A, the corresponding coefficient matrix of linear equations about undetermined coefficients A_1,B_1,A_2,B_2,A_3,B_3 is:

$$C=\begin{pmatrix} 2J_0(1)-7J_1(1) & 2Y_0(1)-7Y_1(1) & 0 & 0 & 0 & 0\\ J_0(4) & Y_0(4) & -J_1(4) & -Y_1(4) & 0 & 0\\ -J_1(4) & -Y_1(4) & -\frac{1}{2}J_1(4)+2J_2(4) & -\frac{1}{2}Y_1(4)+2Y_2(4) & 0 & 0\\ 0 & 0 & J_1(8) & Y_1(8) & -J_2(8) & -Y_2(8)\\ 0 & 0 & \frac{1}{8}J_1(8)-J_2(8) & \frac{1}{8}Y_1(8)-Y_2(8) & -\frac{1}{2}J_2(8)+2J_3(8) & -\frac{1}{2}Y_2(8)+2Y_3(8)\\ 0 & 0 & 0 & 0 & \frac{10}{3}J_2(12)-2J_3(12) & \frac{10}{3}Y_2(12)-2Y_3(12) \end{pmatrix}$$

The row simplest form of matrix C is below：

$$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

It is clear that $R(C)=6=$ numbers of undetermined coefficients. So the boundary value problem of (4.1) has unique solution.