Experimental Simulation of Digital IIR Filter Design Technique Based on Butterworth and Impulse Invariance Concepts

Vorapoj Patanavijit

Assumption University of Thailand
Bangkok, Thailand
e-mail: Patanavijit@yahoo.com

Abstract **- The digital IIR (Infinite Impulse Response) filter is one of the most important parts in telecommunication and Digital Signal Processing (DSP) framework and its design techniques have been investigated for accuracy, computational complexity and practical implementation. In this paper we investigate a novel hybrid IIR filter design technique using Butterworth and impulse invariance concepts, with the implementation objective of less complexity. We investigated two case studies of the low pass IIR filter design techniques using both mathematical modelling and computer simulation. By using Butterworth concept, the magnitude in decibels (dB) and the phase of the frequency response between the analog filter and the digital filter, which are desired by impulse invariance concept, have been comparatively investigated in terms of the design performance and accuracy.**

Keywords - digital IIR (Infinite Impulse Response) filter, Butterworth Filter Desiring Technique, Impulse Invariance Concept, Digital Signal Processing (DSP)

I. INTRODUCTION

In general, a frequency-selective filter is defined as a system [1-5] within the group of LTI (Linear Time-Invariant) systems [6, 7] that passes partial frequency band and completely eliminates all other bands. More general definition of a filter is a system that reforms partial frequency band while completely passing all other bands. Due to its frequency response characteristic, the digital IIR (Infinite Impulse Response) filter [8, 9, 10] has become the primary part of telecommunication and Digital Signal Processing (DSP) framework [13, 14, 15, 16] leading to many design techniques of a digital IIR filter, which have been researched over the last two decades, from technical requirement perspective including accuracy, computational complexity, practical implementation, etc. In this paper, starting from a theoretical perspective, the Butterworth concept [11] is first used to determine the system function of an analog filter. Secondly, the impulsive invariant concept [12] is used to convert from the system function of analog filter to the system function of digital filter. Then we investigate in detail using computer simulation experiments the digital IIR filter design technique of the combined Butterworth and the impulse invariance concepts.

II. THE BUTTERWORTH FILTER

The property of the Butterworth low pass filter [11] is that the magnitude response is flat characteristic (or monotonically decreasing) in both passband and stopband. The magnitude squared response ($\left|H_c(j\Omega)\right|^2$) of this Butterworth low pass filter can be mathematically expressed as following.

$$\left|H_c(j\Omega)\right|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \tag{1}$$

where N is the order of the Butterworth low pass filter and Ω_c is the cutoff frequency (rad/sec)

The design of the Butterworth low pass filter $H_c(j\Omega)$ can be expressed as following step.

1. Determine the order of Butterworth low pass filter N from the specification: R_p (passband ripple parameter) and A_s (stopband attenuation parameter)

$$N = \left\lceil \log\left[\frac{\left(10^{-\frac{A_s}{10}}-1\right)\Big/\left(10^{-\frac{R_p}{10}}-1\right)}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}\right]\right\rceil \tag{2}$$

where

- $\lceil\cdot\rceil$ is the round up operator.
- R_p is the passband ripple parameter (dB) or $R_p = -10\log\left(\frac{1}{1+\varepsilon^2}\right)$ when ε is the passband ripple parameter ($\frac{1}{1+\varepsilon^2} \le \left|H_c(j\Omega)\right|^2 \le 1, \quad |\Omega| \le \Omega_p$).
- A_s is the stopband attenuation parameter (dB) or $A_s = -10\log\left(\frac{1}{A^2}\right)$ when A is the stopband attenuation parameter: ($0 \le \left|H_c(j\Omega)\right|^2 \le \frac{1}{A^2}, \quad \Omega_s \le |\Omega|$).

2. Determine the filter parameter Ω_c (or the cutoff frequency of the Butterworth CT filter) from the specification: N, R_p, A_s, Ω_p and Ω_s.

$$\Omega_c = \frac{\Omega_p}{\sqrt[2N]{\left(10^{-R_p/10}-1\right)}} \tag{3.1}$$

or

$$\Omega_c = \frac{\Omega_s}{\sqrt[2N]{\left(10^{-A_s/10}-1\right)}} \tag{3.2}$$

3. Determine the poles of the system function of the Butterworth CT filter (from the filter parameter N and Ω_c)

$$p_k = \Omega_c \exp\left(\frac{jk\pi}{N}\right) \quad ,k=0,1,2,\ldots,(2N-1) \text{ for } N \text{ is odd.} \tag{4.1}$$

$$p_k = \Omega_c \exp\left(j\left(\frac{\pi}{2N}+\frac{k\pi}{N}\right)\right),k=0,1,2,\ldots,(2N-1) \text{ for } N \text{ is even.} \tag{4.2}$$

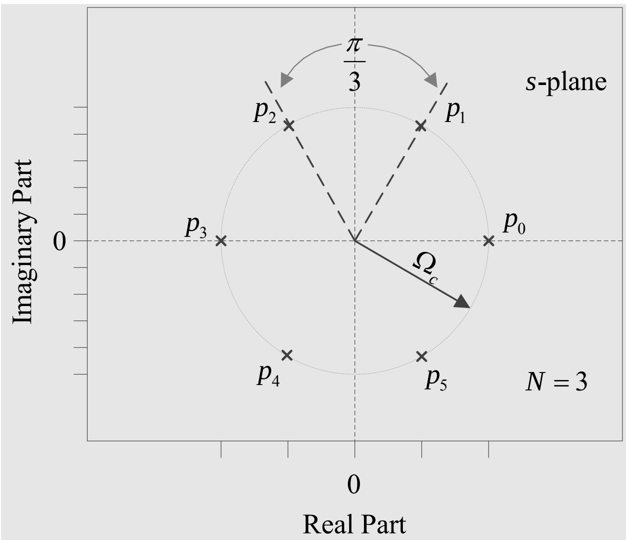

Figure 1. The pole plot of the system function of the Butterworth CT filter for $N=3$

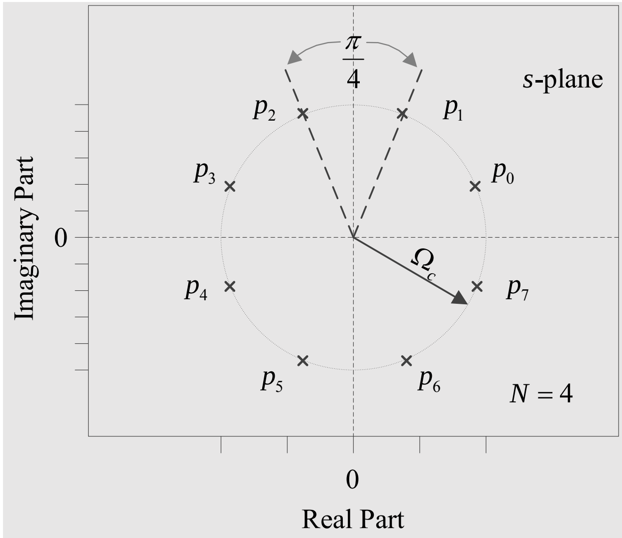

Figure 2. The pole plot of the system function of the Butterworth CT filter for $N=4$

The stable and causal filter $H_c(s)$ can be defined by limiting poles in the left half-plain.

4. Determine the system function ($H_c(s)$) of the Butterworth CT filter (from the estimated poles)

$$H_c(s) = \frac{\Omega_c^N}{\prod_{\text{LHP}}(s-p_k)} \tag{5}$$

III. IMPULSE INVARIANCE

The impulse invariance concept [12] is that the impulse response of the DT system ($h[n]$) is estimated from sampling the impulse response of the CT system ($h_c(t)$). Therefore, the impulse response of the DT system ($h[n]$) can be mathematically expressed as following.

$h[n] = T_d h_c(nT_d)$ where T_d is a sample period. (6)

By using both the CT Fourier analysis and the DT Fourier analysis, the system function of the DT system can be mathematically expressed as following.

$$H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{+\infty} H_c\left(j\frac{\omega}{T_d}+j\frac{2\pi}{T_d}k\right) \tag{7}$$

If the CT filter is bandlimited (or $H_c(j\Omega)=0 \quad ,|\Omega| \geq \pi/T_d$) then the system function of the DT system can be mathematically simplified as follow:

$$H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{+\infty} H_c\left(j\frac{\omega}{T_d}\right) \quad ,|\omega| \leq \pi \tag{8}$$

Therefore, the objective of this impulse invariance concept is for preserving the shape of the impulse response of the CT system ($h_c(t)$) by using the sampling technique. Consequently, the CT system must be bandlimited (or

$H_c(j\Omega) = 0 \quad ,|\Omega| \geq \pi/T_d$) because if the CT system is not bandlimited then the aliasing problem occurs.

The transformation between CT complex plain and DT complex plain, which can be illustrated as following figure, can be mathematically expressed as following.

- The $\text{Re}(s) < 0$ is mapped into $\text{Re}(z) < 1$ (or inside the unit circle).
- The $\text{Re}(s) = 0$ is mapped into $\text{Re}(z) = 1$ (or inside the unit circle).
- The $\text{Re}(s) > 0$ is mapped into $\text{Re}(z) > 1$ (or outside the unit circle).

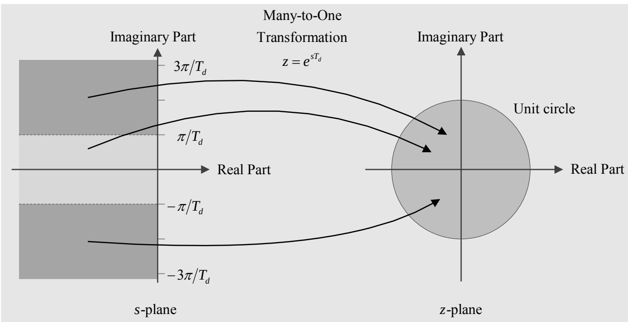

Figure 3. The impulse invariance mapping of the complex plane from the s-plane to z-plane.

The design of the DT IIR lowpass filter by using the impulse invariance can be expressed as following step.

1. Determine the continuous frequency of passband (Ω_p) and stopband (Ω_s) from the specification: ω_p, ω_s and T_d

$$\Omega_p = \frac{\omega_p}{T_d} \quad (9) \quad \text{and} \quad \Omega_s = \frac{\omega_s}{T_d} \quad (10)$$

2. Determine the system function $H_c(s)$ of continuous-time lowpass filter (Butterworth filter, Chebyshev filter or Elliptic filter) from the specification: Ω_p, Ω_s, R_p and A_s (the detail of the continuous-time filter design is expressed in the preceding section)
3. Determine the system function $H_c(s)$ of continuous-time lowpass filter in the partial fraction expansion form, which can be mathematically expressed as following.

$$H_c(s) = \sum_{k=1}^{N} \frac{R_k}{(s - p_k)} \quad (11)$$

4. Determine the system function $H(z)$ of discrete-time lowpass IIR filter from the system function $H_c(s)$ of continuous-time lowpass filter by using impulse invariance transformation for converting the continuous-time poles $\{p_k\}$ to be discrete-time lowpass poles $\{e^{p_k T_d}\}$. Consequently, the system function $H(z)$ of discrete-time lowpass IIR filter can be mathematically expressed as following

$$H(z) = \sum_{k=1}^{N} \frac{R_k}{\left(1 - e^{p_k T_d} z^{-1}\right)} \quad (12)$$

The next section presents numerous examples of the designing of CT Butterworth lowpass filter by using the mathematical analysis for demonstrating the system function, the magnitude response and the phase response for examining the performance of this filter.

IV. EXPERIMENTAL SIMULATION

All simulation outcomes are computed by the MATLAB software, which are operated by PC with CPU: Intel i7-6700HQ and RAM Memory: 16 GB.

A. *Experimental Simulation Results for Case 1*

By using the impulse invariance techniques, design the Butterworth IIR digital filter where the passband gain ($0 \leq |\omega| \leq 0.2\pi$) between 0 dB and -7 dB, and stopband ($0.3\pi \leq |\omega| \leq \pi$) has attenuation of -16 dB where $T_d = 1$. Sketch the magnitude in decibels (dB), the magnitude and the phase of this frequency response of this Butterworth IIR digital filter

The design of the DT IIR lowpass filter by using the impulse invariance can be expressed as following step.

Step 1: Determine the continuous frequency of passband (Ω_p) and stopband (Ω_s) from the specification: ω_p, ω_s and T_d

$$\Omega_p = \frac{\omega_p}{T_d} \quad (9) \quad \rightarrow \quad \Omega_p = \frac{0.2\pi}{1} = 0.2\pi \quad (13.1)$$

and

$$\Omega_s = \frac{\omega_s}{T_d} \quad (10) \quad \rightarrow \quad \Omega_s = \frac{0.3\pi}{1} = 0.3\pi \quad (13.2)$$

Step 2: Determine the system function $H_c(s)$ of continuous-time lowpass filter (Butterworth filter) from the specification: Ω_p, Ω_s, R_p and A_s (the detail of the continuous-time filter design is expressed in the preceding section)

Step 2.1: Determine the filter parameter N (or the order of the Butterworth CT filter). The close form equation for determining the parameter N of the CT Butterworth filter can be mathematically expressed as following.

$$N = \left\lceil \log\left[\frac{\left(10^{-\frac{A_s}{10}} - 1\right) \Big/ \left(10^{-\frac{R_p}{10}} - 1\right)}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}\right] \right\rceil \quad (2)$$

$$N = \left\lceil \log\left[\frac{\left(10^{\frac{-16}{20}}-1\right)\Big/\left(10^{\frac{-7}{20}}-1\right)}{2\log\left(\frac{0.3\pi}{0.2\pi}\right)}\right]\right\rceil$$

$$N = \lceil 2.7986 \rceil \quad \text{or} \quad N = 3 \qquad (13.1)$$

Step 2.2: Determine the filter parameter Ω_c (or the cutoff frequency of the Butterworth CT filter).

By substituting parameter $N=3$ in Eq.(3.1), the filter parameter Ω_c can be mathematically expressed as following.

$$\Omega_c = \frac{\Omega_p}{\sqrt[2N]{\left(10^{-R_p/10}-1\right)}} \qquad (3.1)$$

$$\Omega_c = \frac{0.2\pi}{\sqrt[6]{\left(\left(1/|0.4467|^2\right)-1\right)}} \rightarrow \Omega_c = 0.4985 \qquad (13.4)$$

By substituting parameter $N=3$ in the Eq.(3.2), the filter parameter Ω_c can be mathematically expressed as following.

$$\Omega_c = \frac{\Omega_s}{\sqrt[2N]{\left(10^{-A_s/10}-1\right)}} \qquad (3.2)$$

$$\Omega_c = \frac{0.3\pi}{\sqrt[6]{\left(\left(1/|0.1585|^2\right)-1\right)}} \rightarrow \Omega_c = 0.5122 \qquad (13.5)$$

From the result of the filter parameter Ω_c in the above equation, if the filter parameter $N=3$ then the filter parameter Ω_c can be set between $0.4985 \le \Omega_c \le 0.5122$ therefore the filter parameter Ω_c can be set to be $\Omega_c = 0.5$ in this case for simplicity perspective.

Step 2.3: Determine the poles of the system function of the Butterworth CT filter.

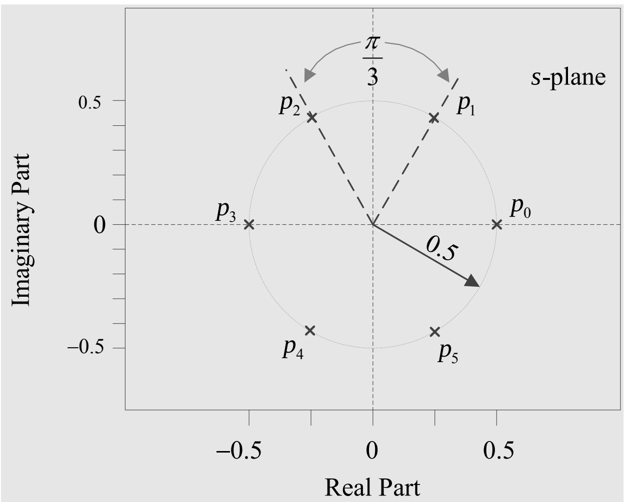

Figure 4. The 6 pole plot of the magnitude of squared function $H_c(s)H_c(-s)$ for 3th order Butterworth filter

From the filter parameter $N=3$ and $\Omega_c = 0.5$, the magnitude of squared function $H_c(s)H_c(-s) = 1\Big/\left(1+\left(s/j\Omega_c\right)^{2N}\right)$ comprises of 6 (or $2N$) poles, which are uniformly located on the circle with 0.5 radius as illustrated in the following figure.

From the stability constrain of the DT-LTI system, the poles of the stable system function $H_c(s)$ must be located in the left half of the s-plain (p_2, p_3 and p_4) as illustrated in the following figure.

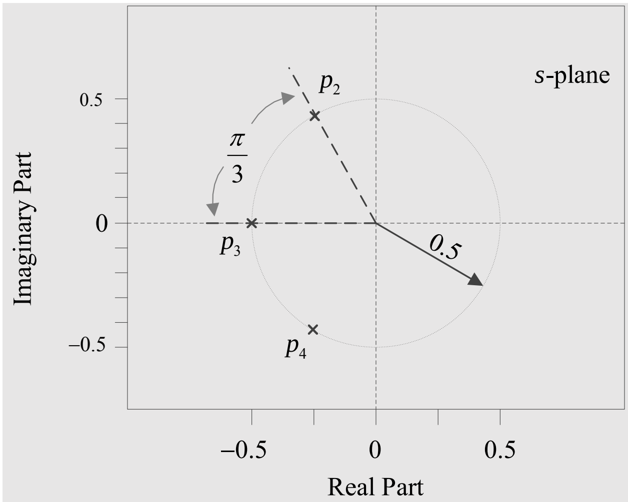

Figure 5. The 3 pole plot of the stable system function $H_c(s)$ for 3th order

From the above figure, the location of each pole can be mathematically expressed as following.

$$(p_3) = \Omega_c\left(-\cos(\pi)\right) \rightarrow (p_3) = 0.5(-1)$$
$$\rightarrow (p_3) = -0.5$$

$$(p_2, p_4) = \Omega_c\left(-\cos(\pi/3) \pm j\sin(\pi/3)\right)$$
$$\rightarrow (p_2, p_4) = 0.5(-0.5 \pm j0.8660)$$
$$\rightarrow (p_2, p_4) = -0.25 \pm j0.4330$$

From the stability constrain of the DT-LTI system, the poles of the stable system function $H_c(s)$ must be located in the left half of the s-plain (p_2, p_3 and p_4) as illustrated in the following figure.

$$\left\{\begin{array}{l} p_2 = (0.5)\exp\left(\frac{j(2)\pi}{3}\right), p_3 = (0.5)\exp\left(\frac{j(3)\pi}{3}\right) \\ , p_4 = (0.5)\exp\left(\frac{j(4)\pi}{3}\right) \end{array}\right\}$$

$$\{p_2 = -0.25+j0.4330, p_3 = -0.5, p_4 = -0.25-j0.4330\} \qquad (13.6)$$

Step 2.4: Determine the system function ($H_c(s)$) of the Butterworth CT filter

$$H_c(s) = \frac{\Omega_c^N}{\prod_{\text{LHP}}(s - p_k)} \quad (5)$$

$$H_c(s) = \frac{\Omega_c^3}{(s - p_2)(s - p_3)(s - p_4)}$$

$$H_c(s) = \frac{(0.5)^3}{\left[(s-(-0.25+j0.4330))(s-(-0.5))(s-(-0.25-j0.4330))\right]}$$

$$H_c(s) = \frac{0.125}{\left[(s+0.25-j0.4330)(s+0.5)(s+0.25+j0.4330)\right]}$$

$$H_c(s) = \frac{0.125}{\left[(s+0.5)(s^2+0.5s+0.25)\right]}$$

$$H_c(s) = \frac{0.125}{(s^3+s^2+0.5s+0.125)} \quad (13.7)$$

Step 3: Determine the system function $H_c(s)$ of continuous-time lowpass filter in the partial fraction expansion form, which can be mathematically expressed as following.

$$H_c(s) = \sum_{k=1}^{N} \frac{R_k}{(s - p_k)} \quad (11)$$

$$H_c(s) = \frac{0.005495}{(s-(-0.5))} - \frac{0.002747-0.03014j}{(s-(-0.25-j0.4330))} + \frac{0.002747+0.03014j}{(s-(-0.25+j0.4330))}$$

Step 4: Determine the system function ($H(z)$) of the DT filter (from the system function ($H_c(s)$) of the Butterworth CT filter) for $T_d = 1$

$$H(z) = \sum_{k=1}^{N} \frac{R_k}{(1 - e^{p_k T_d} z^{-1})} \quad (12)$$

$$H(z) = \left(\frac{0.005495}{(1-e^{(-0.5)}z^{-1})} - \frac{0.002747-0.03014j}{(1-e^{(-0.25-j0.4330)}z^{-1})} - \frac{0.002747+0.03014j}{(1-e^{(-0.25+j0.4330)}z^{-1})} \right)$$

$$H(z) = \left(\frac{0.005495}{(1-0.6065z^{-1})} \frac{0.002747-0.03014j}{(1-(0.7069-0.3268j)z^{-1})} \frac{0.002747+0.03014j}{(1-(0.7069+0.3268j)z^{-1})} \right)$$

$$H(z) = \left(\frac{0.00441-0.0317z^{-1}}{(1-2.0204z^{-1}+1.4641z^{-2}-0.3679z^{-3})} \right) \quad (13.8)$$

First, the magnitude in decibels (dB), the magnitude and the phase of this frequency response of the analog filter $H_c(s)$ can be illustrated as figure 6. Later, By using impulse invariant concept, the magnitude and the phase of this frequency response of the digital filter $H(e^{j\omega})$ can be illustrated as figure 7. From these experimental simulation results, the impulse invariance concept can perfectly converse from the analog filter to the digital filter for magnitude perspective as shown in Fig. 6(a) and Fig. 7(a). However, the phase of the frequency response of the digital filter, which is converted from analog filter, is severely distorted from original analog filter as shown in Fig. 6(c) and Fig. 7(c).

B. *Experimental Simulation Results for Case 2*

By using the impulse invariance techniques, design the Butterworth IIR digital filter where the passband gain ($0 \le |\omega| \le 0.2\pi$) between 0 dB and -1 dB, and stopband ($0.3\pi \le |\omega| \le \pi$) has attenuation of -15 dB where $T_d = 1$. Sketch the magnitude in decibels (dB), the magnitude, the phase and the group delay of this frequency response of this Butterworth IIR digital filter

The design of the DT IIR lowpass filter by using the impulse invariance can be expressed as following step.

Step 1: Determine the continuous frequency of passband (Ω_p) and stopband (Ω_s) from the specification: ω_p, ω_s and T_d

$$\Omega_p = \frac{\omega_p}{T_d} \quad (9) \quad \rightarrow \quad \Omega_p = \frac{0.2\pi}{1} = 0.2\pi \quad (14.1)$$

and

$$\Omega_s = \frac{\omega_s}{T_d} \quad (10) \quad \rightarrow \quad \Omega_s = \frac{0.3\pi}{1} = 0.3\pi \quad (14.2)$$

Step 2: Determine the system function $H_c(s)$ of continuous-time lowpass filter (Butterworth filter) from the specification: Ω_p, Ω_s, R_p and A_s (the detail of the continuous-time filter design is expressed in the preceding section)

Step 2.1: Determine the filter parameter N (or the order of the Butterworth CT filter). The close form equation for determining the parameter N of the CT Butterworth filter can be mathematically expressed as following.

$$N = \left\lceil \log\left[\frac{\left(10^{-\frac{A_s}{10}} - 1\right) \Big/ \left(10^{-\frac{R_p}{10}} - 1\right)}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)} \right] \right\rceil \quad (2)$$

$$N = \left\lceil \log\left[\frac{\left(10^{\frac{-15}{20}} - 1\right) \Big/ \left(10^{\frac{-1}{20}} - 1\right)}{2\log\left(\frac{0.3\pi}{0.2\pi}\right)} \right] \right\rceil$$

$$N = \lceil 5.8857 \rceil \quad \text{or} \quad N = 6 \quad (14.3)$$

Step 2.2: Determine the filter parameter Ω_c (or the cutoff frequency of the Butterworth CT filter).

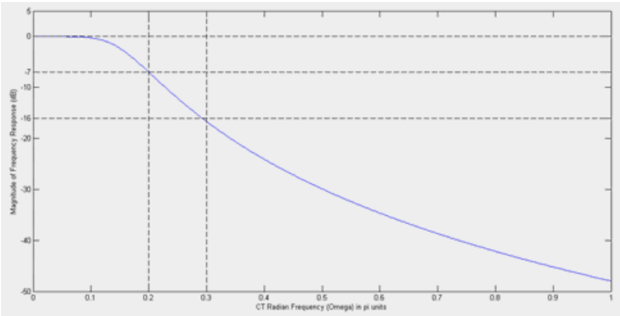

Figure 6 (a) The relationship between the magnitude in decibels (dB) of the frequency response of the analog filter, $20\log_{10}\left|H\left(e^{j\omega}\right)\right|$, and Ω.

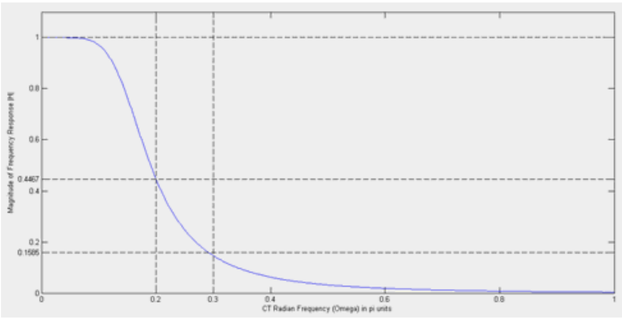

Figure 6 (b) The relationship between the magnitude of the frequency response of the analog filter, $\left|H_c\left(j\Omega\right)\right|$, and analog frequency Ω.

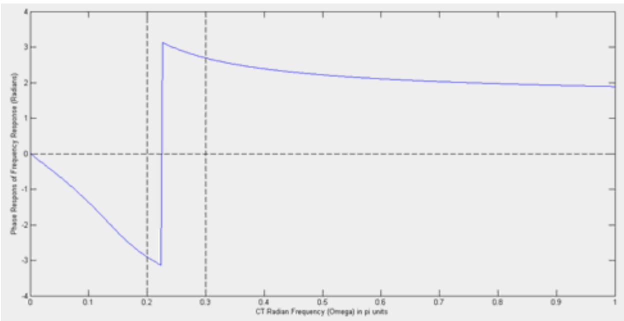

Figure 6 (c) The relationship between the phase of the frequency response of the analog filter, $\angle H_c\left(j\Omega\right)$, and analog frequency Ω.

By substituting parameter $N = 6$ in Eq.(3.1), the filter parameter Ω_c can be mathematically expressed as following.

$$\Omega_c = \frac{\Omega_p}{\sqrt[2N]{\left(10^{-R_p/10} - 1\right)}} \tag{3.1}$$

$$\Omega_c = \frac{2\pi}{\sqrt[2(6)]{\left(10^{-(-1/10)} - 1\right)}} \quad \rightarrow \quad \Omega_c = 0.7032 \tag{14.4}$$

By substituting parameter $N = 6$ in the Eq.(3.2), the filter parameter Ω_c can be mathematically expressed as following.

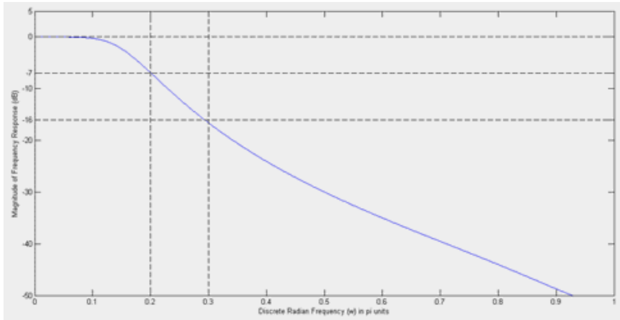

Figure 7 (a) The relationship between the magnitude in decibels (dB) of the frequency response of the digital filter, $20\log_{10}\left|H\left(e^{j\omega}\right)\right|$, and ω.

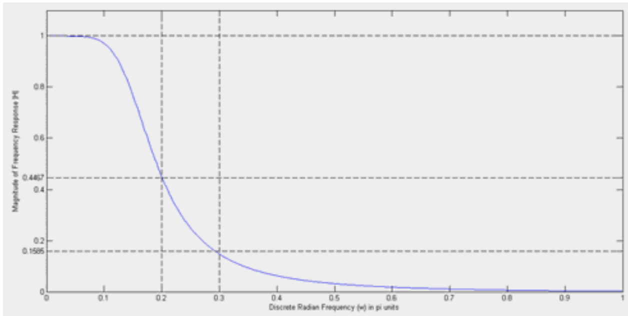

Figure 7 (b) The relationship between the magnitude of the frequency response of the digital filter, $\left|H\left(e^{j\omega}\right)\right|$, and digital frequency ω.

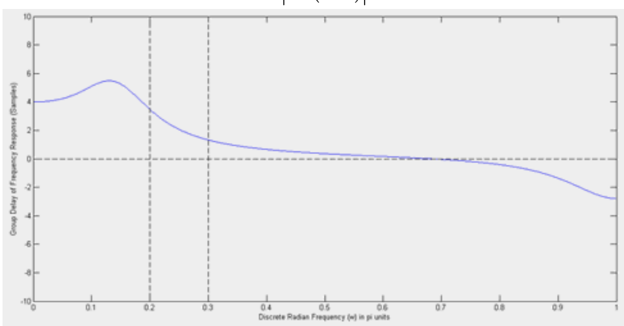

Figure 7 (c) The relationship between the phase of the frequency response of the digital filter, $\angle H\left(e^{j\omega}\right)$, and digital frequency ω.

$$\Omega_c = \frac{\Omega_s}{\sqrt[2N]{\left(10^{-A_s/10} - 1\right)}} \tag{3.2}$$

$$\Omega_c = \frac{3\pi}{\sqrt[2(6)]{\left(10^{-(-15/10)} - 1\right)}} \quad \rightarrow \quad \Omega_c = 0.7087 \tag{14.4}$$

From the result of the filter parameter Ω_c in the above equation, if the filter parameter $N = 6$ then the filter parameter Ω_c can be set between $0.7032 \le \Omega_c \le 0.7087$ therefore, the filter parameter Ω_c can be set to be $\Omega_c = 0.7032$ in this case.

Step 2.3: Determine the poles of the system function of the Butterworth CT filter.

From the filter parameter $N = 6$ and $\Omega_c = 0.7032$, the magnitude of squared function:

DOI 10.5013/IJSSST.a.19.06.40

ISSN: 1473-804x online, 1473-8031 print

$H_c(s)H_c(-s) = 1/\left(1+(s/j\Omega_c)^{2N}\right)$ which comprises 12 (or $2N$) poles, which are uniformly located on the circle with $\Omega_c = 0.7032$ radius as illustrated in the following figure. From the stability constrain of the DT-LTI system, the poles of the stable system function $H_c(s)$ must be located in the left half of the s-plain (p_3, p_4, p_5, p_6, p_7 and p_8) as illustrated in the following figure.

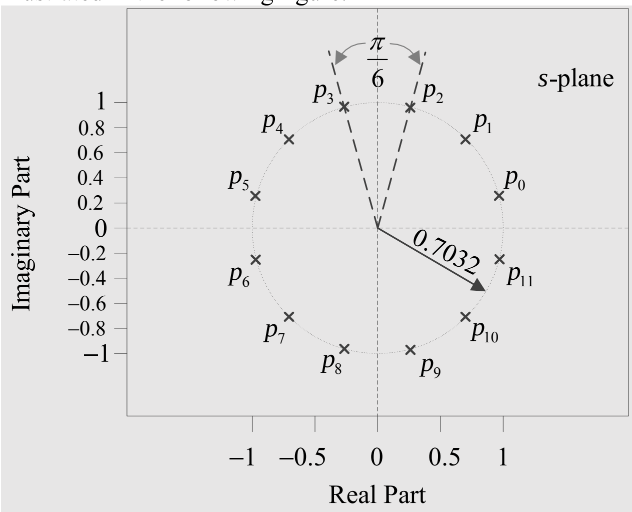

Figure 8. The 12 pole plot of the magnitude of squared function $H_c(s)H_c(-s)$ for 6th order Butterworth filter

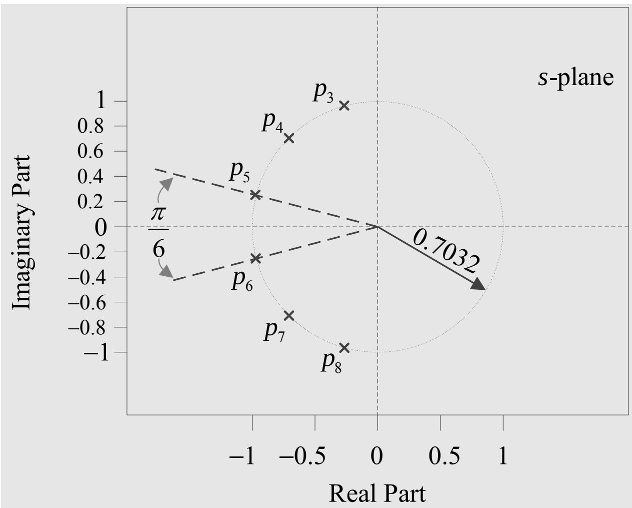

Figure 9. The 6 pole plot of the stable system function $H_c(s)$ for 6th order

From the above figure, the location of each pole can be mathematically expressed as following.

$$(p_3, p_8) = \Omega_c\left(-\cos(5\pi/12) \pm j\sin(5\pi/12)\right)$$

$$\rightarrow \quad (p_3, p_8) = 0.7032(-0.2588 \pm j0.9659)$$

$$\rightarrow \quad (p_3, p_8) = -0.1820 \pm j0.6792 \tag{14.6}$$

$$(p_4, p_7) = \Omega_c\left(-\cos(3\pi/12) \pm j\sin(3\pi/12)\right)$$

$$\rightarrow \quad (p_4, p_7) = 0.7032(-0.7071 \pm j0.7071)$$

$$\rightarrow \quad (p_4, p_7) = -0.4972 \pm j0.4972 \tag{14.7}$$

$$(p_5, p_6) = \Omega_c\left(-\cos(\pi/12) \pm j\sin(\pi/12)\right)$$

$$\rightarrow \quad (p_5, p_6) = 0.7032(-0.9659 \pm j0.2588)$$

$$\rightarrow \quad (p_5, p_6) = -0.6792 \pm j0.1820 \tag{14.8}$$

From the stability constrain of the DT-LTI system, the poles of the stable system function $H_c(s)$ must be located in the left half of the s-plain (p_2, p_3 and p_4) as illustrated in the following figure.

$$\left\{ \begin{array}{l} p_2 = (0.5)\exp\left(\dfrac{j(2)\pi}{3}\right), p_3 = (0.5)\exp\left(\dfrac{j(3)\pi}{3}\right) \\ , p_4 = (0.5)\exp\left(\dfrac{j(4)\pi}{3}\right) \end{array} \right\}$$

$$\{p_2 = -0.25 + j0.4330, p_3 = -0.5, p_4 = -0.25 - j0.4330\}$$

Step 2.4: Determine the system function ($H_c(s)$) of the Butterworth CT filter

$$H_c(s) = \frac{\Omega_c^N}{\prod_{\text{LHP}}(s - p_k)} \tag{5}$$

$$H_c(s) = \frac{\Omega_c^6}{(s-p_3)(s-p_4)(s-p_5)(s-p_6)(s-p_7)(s-p_8)}$$

$$H_c(s) = \frac{(0.7032)^6}{\left[\begin{array}{l} \left(s - \begin{pmatrix} -0.1820 \\ +j0.6792 \end{pmatrix}\right)\left(s - \begin{pmatrix} -0.4972 \\ +j0.4972 \end{pmatrix}\right)\left(s - \begin{pmatrix} -0.6792 \\ +j0.1820 \end{pmatrix}\right) \\ \times\left(s - \begin{pmatrix} -0.6792 \\ -j0.1820 \end{pmatrix}\right)\left(s - \begin{pmatrix} -0.4972 \\ -j0.4972 \end{pmatrix}\right)\left(s - \begin{pmatrix} -0.1820 \\ -j0.6792 \end{pmatrix}\right) \end{array}\right]}$$

$$H_c(s) = \frac{0.1209}{\left[\begin{array}{l} \left[\left(s - \begin{pmatrix} -0.1820 \\ +j0.6792 \end{pmatrix}\right)\left(s - \begin{pmatrix} -0.1820 \\ -j0.6792 \end{pmatrix}\right)\right] \\ \times\left[\left(s - \begin{pmatrix} -0.4972 \\ +j0.4972 \end{pmatrix}\right)\left(s - \begin{pmatrix} -0.4972 \\ -j0.4972 \end{pmatrix}\right)\right] \\ \times\left[\left(s - \begin{pmatrix} -0.6792 \\ +j0.1820 \end{pmatrix}\right)\left(s - \begin{pmatrix} -0.6792 \\ -j0.1820 \end{pmatrix}\right)\right] \end{array}\right]}$$

$$H_c(s) = \frac{0.1209}{\left[\begin{array}{l} (s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945) \\ (s^2 + 1.3585s + 0.4945) \end{array}\right]}$$

$$H_c(s) = \frac{0.1209}{s^6 + 2.7168s^5 + 3.6905s^4 + 3.1782s^3 + 1.8247s^2 + 0.6642s + 0.1209} \tag{14.9}$$

DOI 10.5013/IJSSST.a.19.06.40 ISSN: 1473-804x online, 1473-8031 print

Step 3: Determine the system function $H_c(s)$ of continuous-time lowpass filter in the partial fraction expansion form, which can be mathematically expressed as following.

$$H_c(s)=\sum_{k=1}^{N}\frac{R_k}{(s-p_k)} \tag{11}$$

$$H_c(s)=\frac{0.1209}{\left[\begin{array}{l}\left[\left(s-\left(\begin{array}{c}-0.1820\\+j0.6792\end{array}\right)\right)\left(s-\left(\begin{array}{c}-0.1820\\-j0.6792\end{array}\right)\right)\right]\\ \left[\left(s-\left(\begin{array}{c}-0.4972\\+j0.4972\end{array}\right)\right)\left(s-\left(\begin{array}{c}-0.4972\\-j0.4972\end{array}\right)\right)\right]\\ \times\left[\left(s-\left(\begin{array}{c}-0.6792\\+j0.1820\end{array}\right)\right)\left(s-\left(\begin{array}{c}-0.6792\\-j0.1820\end{array}\right)\right)\right]\end{array}\right]}$$

$$H_c(s)=\left(\begin{array}{l}\frac{0.1463+j0.2487}{(s-(-0.1820+j0.6792))}+\frac{0.1463-j0.2487}{(s-(-0.1820-j0.6792))}\\ +\frac{1.0720+j0.00003}{(s-(-0.4972+j0.4972))}+\frac{1.0720-j0.00003}{(s-(-0.4972-j0.4972))}\\ +\frac{0.9283-j1.6077}{(s-(-0.6792+j0.1820))}+\frac{0.9283+j1.6077}{(s-(-0.6792-j0.1820))}\end{array}\right)$$

Step 4: Determine the system function ($H(z)$) of the DT filter (from the system function ($H_c(s)$) of the Butterworth CT filter) for $T_d=1$

$$H(z)=\sum_{k=1}^{N}\frac{R_k}{\left(1-e^{p_kT_d}z^{-1}\right)} \tag{12}$$

$$H(z)=\left(\begin{array}{l}\frac{0.1463+j0.2487}{\left(1-e^{(-0.1820+j0.6792)}z^{-1}\right)}+\frac{0.1463-j0.2487}{\left(1-e^{(-0.1820-j0.6792)}z^{-1}\right)}+\frac{1.0720+j0.00003}{\left(1-e^{(-0.4972+j0.4972)}z^{-1}\right)}\\ +\frac{1.0720-j0.00003}{\left(1-e^{(-0.4972-j0.4972)}z^{-1}\right)}+\frac{0.9283-j1.6077}{\left(1-e^{(-0.6792+j0.1820)}z^{-1}\right)}+\frac{0.9283+j1.6077}{\left(1-e^{(-0.6792-j0.1820)}z^{-1}\right)}\end{array}\right)$$

$$H(z)=\left(\begin{array}{l}\left[\frac{0.1463+j0.2487}{\left(1-(0.6486+j0.5236)z^{-1}\right)}+\frac{0.1463-j0.2487}{\left(1-(0.6486-j0.5236)z^{-1}\right)}\right]\\ +\left[\frac{1.0720+j0.00003}{\left(1-(0.5346+j0.2901)z^{-1}\right)}+\frac{1.0720-j0.00003}{\left(1-(0.5346-j0.2901)z^{-1}\right)}\right]\\ +\left[\frac{0.9283-j1.6077}{\left(1-(0.4986+j0.0918)z^{-1}\right)}+\frac{0.9283+j1.6077}{\left(1-(0.4986-j0.0918)z^{-1}\right)}\right]\end{array}\right)$$

$$H(z)=\left\{\begin{array}{l}\left[\frac{0.2871-0.4466z^{-1}}{\left(1-1.2971z^{-1}+0.6949z^{-2}\right)}\right]+\left[\frac{-2.1428+1.1455z^{-1}}{\left(1-1.0691z^{-1}+0.3699z^{-2}\right)}\right]\\ +\left[\frac{1.8557+0.6303z^{-1}}{\left(1-0.9972z^{-1}+0.2570z^{-2}\right)}\right]\end{array}\right. \tag{13}$$

First, the magnitude in decibels (dB), the magnitude and the phase of this frequency response of the analog filter $H_c(s)$ can be illustrated as figure 10. Later, the magnitude and the phase of this frequency response of the digital filter $H(e^{j\omega})$ can be illustrated as figure 11. From these experimental simulation results, the impulse invariance concept can perfectly converse from the analog filter to the digital filter for magnitude perspective as shown in Fig. 10(a) and Fig. 11(a). However, the phase of the frequency response of the digital filter, which is converted from analog filter, is severely distorted from original analog filter as shown in Fig. 10(c) and Fig. 11(c).

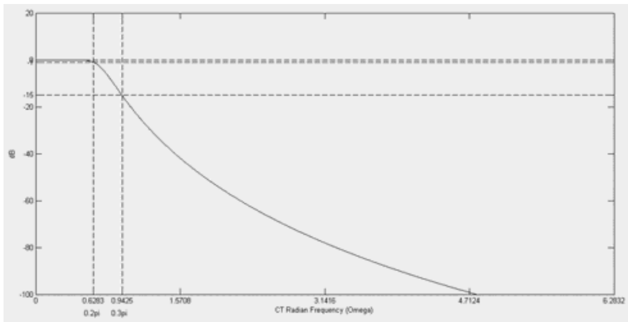

Figure 10 (a) The relationship between the magnitude in decibels (dB) of the frequency response of the analog filter, $20\log_{10}\left|H\left(e^{j\omega}\right)\right|$, and Ω.

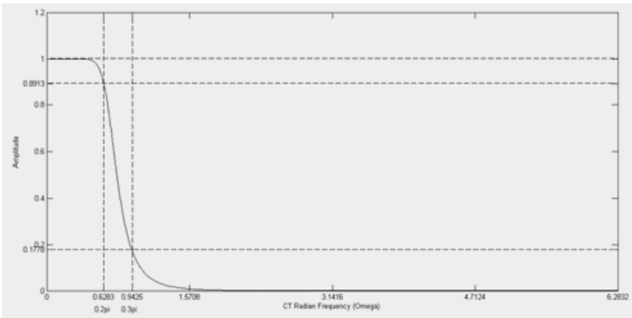

Figure 10 (b) The relationship between the magnitude of the frequency response of the analog filter, $\left|H_c(j\Omega)\right|$, and analog frequency Ω.

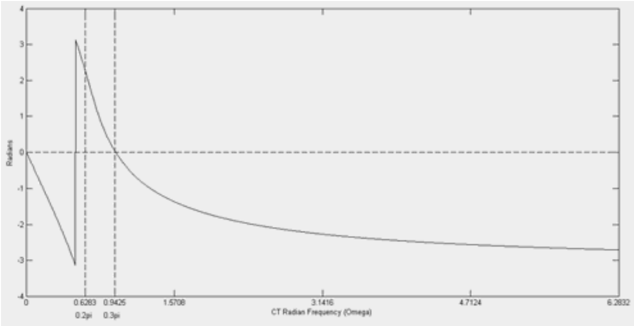

Figure 10 (c) The relationship between the phase of the frequency response of the analog filter, $\angle H_c\left(j\Omega\right)$, and analog frequency Ω.

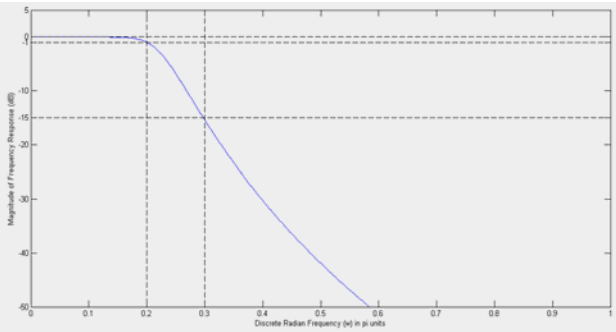

Figure 11 (a) The relationship between the magnitude in decibels (dB) of the frequency response of the digital filter, $20\log_{10}\left|H\left(e^{j\omega}\right)\right|$, and ω.

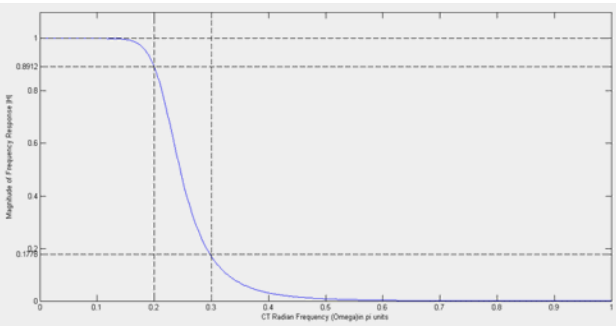

Figure 11 (b) The relationship between the magnitude of the frequency response of the digital filter, $\left|H\left(e^{j\omega}\right)\right|$, and digital frequency ω.

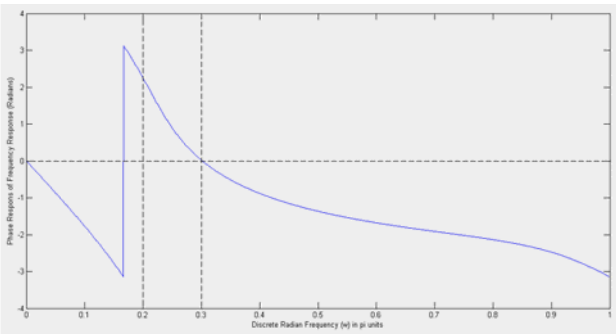

Figure 11 (c) The relationship between the phase of the frequency response of the digital filter, $\angle H\left(e^{j\omega}\right)$, and digital frequency ω.

V. CONCLUSION

We investigated the capacity of the digital IIR filter design technique using Butterworth and impulse invariance concept. We demonstrated the filter desired properties in both mathematical and computer simulation. The results show the design technique has good performance for magnitude response requirements but poor performance for phase response requirements, especially at high frequency.

ACKNOWLEDGMENT

The research project was funded by Assumption University.

REFERENCES

[1] A. V. Oppenhiem, A. S. Willsky, and S. H. Nawab, “Signals and Systems”, Prentice-Hall, 2nd ed., 1997.

[2] A. V. Oppenheim, and R.W. Schafer, “Discrete-Time Signal Processing”, Prentice Hall, 2nd ed., 1998.

[3] A. V. Oppenheim, and R.W. Schafer, “Discrete-Time Signal Processing”, Prentice Hall, 3nd ed., 2010.

[4] C. Ray Wylie & Louis C. Barrett, “Advanced Engineering Mathematics”, McGraw-Hill Companies, Inc., 6th ed., 1995

[5] Erwin Kreyszig, “Advanced Engineering Mathematics”, John Wiley & Sons, Inc., 10th ed., 2011.

[6] Hwei P. Hsu, “SCHAUM'S OUTLINES OF Theory and Problems of Signals and Systems” SCHAUM's Outlines, McGraw-Hill Companies, Inc., 1995.

[7] L. Phillips, J. M. Parr, and E. A. Riskin, “Signals, Systems, and Transforms”, Prentice-Hall, 4th ed., 2007.

[8] Monson H. Hayes, "Schaum's Outline of Theory and Problems of Digital Signal Processing" SCHAUM's Outlines , McGraw-Hill Companies, Inc., 1999

[9] Rafael C. Gonzalez & Richard E. Woods, “Digital Image Processing”, Prentice-Hall, 3th ed., 2010.

[10] S. Haykin and B. V. Veen, “Signals and Systems”, John Wiley & Sons, Inc., 2nd ed., 2003.

[11] Shibendu Mahata, Suman Kumar Saha, Rajib Kar, Durbadal Mandal, Accurate integer-order rational approximation of fractional-order low-pass Butterworth filter using a metaheuristic optimisation approach, IET Signal Processing, 2018.

[12] T.J. Cavicchi, Impulse invariance and multiple-order poles, IEEE Transactions on Signal Processing, Volume: 44 , Issue: 9, 1996

[13] Vinay K. Ingle and John G. Proakis, “Digital Signal Processing using Matlab”, Brooks/Cole Thomson Learning, 2000.

[14] Vorapoj Patanavijit and Kornkamol Thakulsukanant, “Mathematical Tutorial of Discrete-Time Analysis of Aliasing and Non-Aliasing Periodic Sampling Concept with Fourier Analysis for Digital Signal Processing and Digital Communication Prospective”, SDU Research Journal Sciences and Technology, Suan Dusit Rajabhat University, Vol. 9, No. 3, Sep.-Dec. 2016. (indexed by TCI Group 1 and ACI)

[15] Vorapoj Patanavijit, “Mathematical Tutorial of Discrete-Time Analysis of Sampling Rate Changing Concept for Digital Signal Processing and Digital Communication Prospective”, RMUTT Journal Sciences and Technology, RMUTT, Vol. 6, No. 2 , July.-Dec. 2017. (indexed by TCI Group 1 and ACI)

[16] Vorapoj Patanavijit, “Conceptual Framework of Super Resolution Reconstruction Based on Frequency Domain From Aliased Multi-Low Resolution Images: Theory Part”, Panyapiwat Journal, Panyapiwat Institute of Management (PIM), Thailand, Vol. 8, No. 2, May. – Aug. 2016. (indexed by TCI Group 1 and ACI).