Computer Modelling and Simulation of Bilinear Transformation on the Digital Infinite Impulse Response Butterworth Filter

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Abstract – **During the past two and half decades digital telecommunication technology and digital signal processing models have seen accelerated development and implementation to support the electronic infrastructure of the communication industry. The digital IIR (Infinite Impulse Response) filter has been the focus as the primary component within this area with dramatic progress in term of design, precision, fast operation and easy implementation. Our paper here systematically orders the significance of the digital IIR filter design approach based on Butterworth filtering and bilinear transformation for both statistical and simulation models computation. First, the magnitude response and the phase response of the transfer function of the analog filter are computed by Butterworth filtering approach. Next, the magnitude response and the phase response of the transfer function of the digital filter are computed by bilinear transformation. The simulation results of the analog filter and the digital filter are computationally analyzed in term of performance.**

Keywords - digital IIR (Infinite Impulse Response) filter, Butterworth Filter Desiring Technique, Bilinear Transform, Digital Signal Processing (DSP)

I. INTRODUCTION AND LITERATURE REVIEW

Traditionally, a system [1-5] which sends some fragment frequency collection but suppresses all other frequency collection, is designated as band-pass filter or frequency-selective filter, which is one of the most well-known and applicable set of LTI (Linear Time-Invariant) systems [6,7]. However, a system which alters fragment frequency collection and sends all other frequency collection, can be designated as the frequency-selective filter in broader sense. Because of the required transfer function characteristics, the digital filter based on IIR (Infinite Impulse Response) [8, 9, 10] is one of the primary components in digital telecommunication and digital signal processing [13, 14, 15, 16] where many design approaches of a digital IIR filter have developed for over the last two and half decades in term of design, precision, fast operation, easy implementation, etc. From algebraic point of view, the Butterworth filter approach [11, 17] has been initially used to design the analog filter with the required transfer function because this approach is simple and widely understood and accepted. Subsequently, the transfer function of analog filter is converted to the transfer function of digital filter by using the bilinear transform [12]. Thus, this paper undertake to analyze the digital IIR filter design approach based on Butterworth filtering model and bilinear transformation for both statistical computation and simulation computation.

II. THE BUTTERWORTH FILTER MODEL

For the statistical property of Butterworth low-pass filter [11,17], the magnitude response in both stopband and passband is smooth attribute (or monotonically declining). The magnitude squared response ($\left|H_c\left(j\Omega\right)\right|^2$) of this Butterworth low-pass filter can be algebraic written as afterward.

$$\left|H_c\left(j\Omega\right)\right|^2 = \left(1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right)^{-1} \tag{1}$$

where N is the order of the Butterworth low-pass filter and Ω_c is the low-pass cutoff frequency (rad/sec)

The arrangement of the Butterworth low-pass filter can be expressed as following steps:

-1. Calculate the order of Butterworth low-pass filter N from the specification: R_p (passband ripple parameter) and A_s (stopband attenuation parameter)

$$N = \left\lceil \log\left[\left(\left(10^{-\frac{A_s}{10}}-1\right)\Big/\left(10^{-\frac{R_p}{10}}-1\right)\right)\Big/ 2\log\left(\frac{\Omega_s}{\Omega_p}\right)\right]\right\rceil \tag{2}$$

where:

- $\lceil \cdot \rceil$ is the round up operator.
- R_p is the passband ripple parameter (dB).
- A_s is the stopband attenuation parameter (dB).

-2. Calculate the filter parameter Ω_c (or the cutoff frequency of the Butterworth CT filter) from the specification: N, R_p, A_s, Ω_p and Ω_s.

$$\Omega_c = \Omega_p \Big/ \sqrt[2N]{\left(10^{-R_p/10} - 1\right)} \tag{3.1}$$

or

$$\Omega_c = \Omega_s \Big/ \sqrt[2N]{\left(10^{-A_s/10} - 1\right)} \tag{3.2}$$

-3. Calculate the poles of the transfer function of the Butterworth CT filter (from the filter parameter N and Ω_c)

$$p_k = \Omega_c \exp\left(\frac{jk\pi}{N}\right) \quad ,k = 0,1,2,\ldots,(2N-1) \text{ for } N \text{ is odd.} \tag{4.1}$$

$$p_k = \Omega_c \exp\left(j\left(\frac{\pi}{2N} + \frac{k\pi}{N}\right)\right), k = 0,1,2,\ldots,(2N-1) \text{ for } N \text{ is even.} \tag{4.2}$$

The stable and causal filter $H_c(s)$ can be defined by limiting poles in the left half-plain.

-4. Calculate the transfer function ($H_c(s)$) of the Butterworth CT filter

$$H_c(s) = \frac{\Omega_c^N}{\prod_{\text{LHP}} (s - p_k)} \tag{5}$$

III. BILINEAR TRANSFORMATION MODELS

The bilinear transformation concept is a nonlinear mapping that converts the continuous–time variable (s) of the s-plane to the discrete-time variable (z) of the z-plane. Therefore, the mathematical relationship between the continuous-time variable s and the discrete-time variable z can be algebraic written as follows:

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \text{ where } T_d \text{ is a sample period.} \tag{6}$$

Therefore, the DT transfer function ($H(z)$) can be mathematically expressed as:

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \tag{7}$$

The mathematical relationship between the continuous–time frequency ($-\infty \le \Omega \le +\infty$) of the s-plane to the discrete-time frequency ($-\infty \le \omega \le +\infty$) of the z-plane can be algebraic written as:

$$\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \quad \text{where } \Omega \text{ is the analog frequency and } \omega \text{ is the digital frequency} \tag{8}$$

or

$$\omega = 2\arctan\left(\frac{\Omega T_d}{2}\right) \tag{9}$$

The transformation between CT complex plain and DT complex plain, which can be illustrated as following figure, can be algebraic written as afterward.

- The $\text{Re}(s) < 0$ is mapped into $\text{Re}(z) < 1$ (or inside the unit circle).
- The $\text{Re}(s) = 0$ is mapped into $\text{Re}(z) = 1$ (or inside the unit circle).
- The $\text{Re}(s) > 0$ is mapped into $\text{Re}(z) > 1$ (or outside the unit circle).

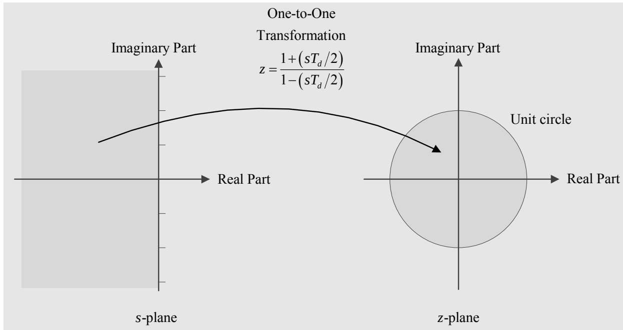

Figure 1. The bilinear transformation mapping of the complex plane from the s-plane to z-plane.

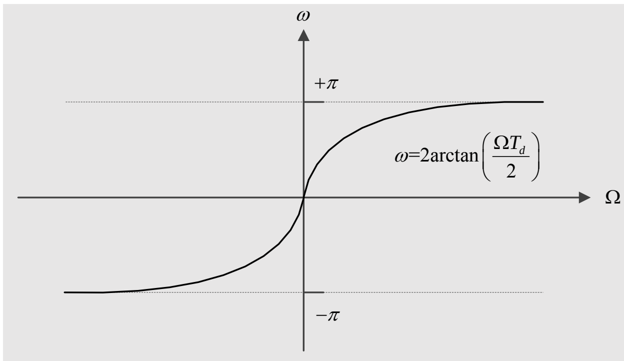

Figure 2. The bilinear transformation mapping of the continuous–time frequency ($-\infty \le \Omega \le +\infty$) to the discrete-time frequency ($-\infty \le \omega \le +\infty$).

The design of the DT IIR low-pass filter by using the bilinear transformation can be algebraically written as steps as follows:

-1. Calculate the continuous frequency of passband (Ω_p) and stopband (Ω_s) from the specification: ω_p, ω_s and T_d

$$\Omega_p = \frac{2}{T_d}\tan\left(\frac{\omega_p}{2}\right) \tag{10}$$

and

$$\Omega_s = \frac{2}{T_d}\tan\left(\frac{\omega_s}{2}\right) \tag{11}$$

-2. Calculate the transfer function $H_c(s)$ of continuous-time low-pass filter (Butterworth filter) from the specification: Ω_p, Ω_s, R_p and A_s (the detail of the continuous-time filter design is expressed in the preceding section).
-3. Calculate the transfer function $H(z)$ of discrete-time low-pass IIR filter from the transfer function $H_c(s)$ of continuous-time low-pass filter by using bilinear, which can be algebraic written as:

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \tag{12}$$

The later part presents numerous experimental cases of the designing of CT Butterworth lowpass filter by using bilinear transform for determining the transfer function and, next, the magnitude response and the phase response of the transfer function is illustrated for examining the performance of this filter.

IV. SIMULATION RESULTS

In this section, all computer results are simulated by MATLAB software and executed by PC at specification: CPU Intel i7-6700HQ and RAM Memory: 16 GB.

A. Computer Results for Case 1

By using the bilinear transformation concept, design the Butterworth IIR digital filter where the passband gain ($0 \le |\omega| \le 0.2\pi$) between 0 dB and -7 dB, and stopband ($0.3\pi \le |\omega| \le \pi$) has attenuation of -16 dB where $T_d = 1$. Sketch the magnitude in decibels (dB), the magnitude, the phase and the group delay of this frequency response of this Butterworth IIR digital filter

The design of the DT IIR low-pass filter by using the bilinear transformation can be expressed in the following steps:

Step 1: Calculate the continuous frequency of passband (Ω_p) and stopband (Ω_s) from the specification: ω_p, ω_s and T_d

$$\Omega_p = \frac{2}{T_d}\tan\left(\frac{\omega_p}{2}\right)(10) \rightarrow \Omega_p = \frac{2}{(1)}\tan\left(\frac{0.2\pi}{2}\right) = 0.6498 \tag{13.1}$$

and

$$\Omega_s = \frac{2}{T_d}\tan\left(\frac{\omega_s}{2}\right)(11) \rightarrow \Omega_s = \frac{2}{(1)}\tan\left(\frac{0.3\pi}{2}\right) = 1.0191 \tag{13.2}$$

Step 2: Calculate the transfer function $H_c(s)$ of continuous-time low-pass filter (Butterworth filter) [17] from the specification: Ω_p, Ω_s, R_p and A_s for bilinear transformation concept (the detail of the continuous-time filter design is expressed in the preceding section):

$$H_c(s) = \frac{0.1370}{\left(s^3 + 1.0310s^2 + 0.5315s + 0.1370\right)} \tag{13.3}$$

Step 3: Calculate the transfer function ($H(z)$) of the DT filter (from the transfer function ($H_c(s)$) of the Butterworth CT filter) for $T_d = 1$ by using bilinear transformation:

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \tag{12}$$

$$H(z) = H_c\left(\frac{2}{(1)}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$$

$$H(z) = H_c\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$$

$$H(z) = \frac{0.1370}{\left(\begin{array}{l}\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^3 + 1.0310\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2 \\ +0.5315\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 0.1370\end{array}\right)}$$

$$H(z) = \frac{0.0103 + 0.0308z^{-1} + 0.0308z^{-2} + 0.0103z^{-3}}{1 - 2.0002z^{-1} + 1.4428z^{-2} - 0.3604z^{-3}} \tag{13.4}$$

First, the magnitude in decibels (dB), the magnitude and the phase of this frequency response of the analog filter $H_c(s)$ [17] can be illustrated as figure 3. Later, the magnitude and the phase of this frequency response of the digital filter $H(e^{j\omega})$ can be illustrated as figure 4. From these computer results, the bilinear transformation concept can perfectly converse from the analog filter to the digital filter for magnitude perspective as shown in Fig. 3(a) and Fig. 4(a). Moreover, the phase of the frequency response of the digital filter, which is converted from analog filter, is slightly distorted from original analog filter as shown in Fig. 3(c) and Fig. 4(c).

B. Computer Results for Case 2

By using the bilinear transformation concept, design the Butterworth IIR digital filter where the passband gain ($0 \le |\omega| \le 0.2\pi$) between 0 dB and -1 dB, and stopband ($0.3\pi \le |\omega| \le \pi$) has attenuation of -15 dB where $T_d = 1$. Sketch the magnitude in decibels (dB), the magnitude, the phase and the group delay of this frequency response of this Butterworth IIR digital filter

The design of the DT IIR low-pass filter by using the bilinear transformation can be expressed as following step.

Step 1: Determine the continuous frequency of passband (Ω_p) and stopband (Ω_s) from the specification: ω_p, ω_s and T_d

$$\Omega_p = \frac{2}{T_d}\tan\left(\frac{\omega_p}{2}\right)(10) \rightarrow \Omega_p = \frac{2}{(1)}\tan\left(\frac{0.2\pi}{2}\right) = 0.6498 \quad (14.1)$$

and

$$\Omega_s = \frac{2}{T_d}\tan\left(\frac{\omega_s}{2}\right)(11) \rightarrow \Omega_s = \frac{2}{(1)}\tan\left(\frac{0.3\pi}{2}\right) = 1.0191 \quad (14.2)$$

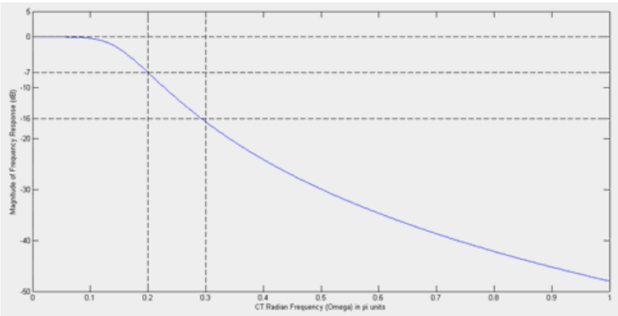

Figure 3 (a) The relationship between the magnitude in decibels (dB) of the frequency response of the analog filter, $20\log_{10}\left|H\left(e^{j\omega}\right)\right|$, and Ω.

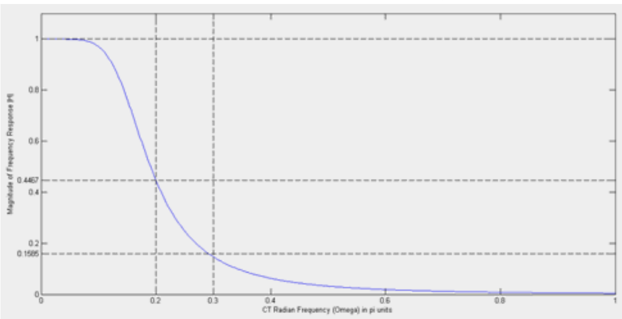

Figure 3 (b) The relationship between the magnitude of the frequency response of the analog filter, $\left|H_c\left(j\Omega\right)\right|$, and analog frequency Ω.

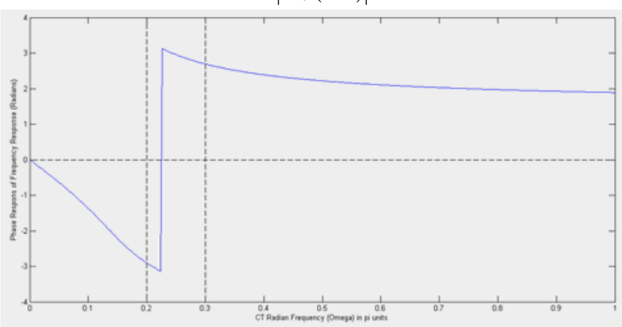

Figure 3 (c) The relationship between the phase of the frequency response of the analog filter, $\angle H_c\left(j\Omega\right)$, and analog frequency Ω.

Step 2: Calculate the transfer function $H_c(s)$ of continuous-time low-pass filter (Butterworth filter) [17] from the specification: Ω_p, Ω_s, R_p and A_s for bilinear transformation concept (the detail of the continuous-time filter design is expressed in the preceding section):

$$H_c(s) = \frac{0.1480}{\left(\begin{array}{l} s^6 + 2.8100s^5 + 3.9482s^4 + 3.5168s^3 \\ +2.0884s^2 + 0.7862s + 0.1480 \end{array}\right)} \quad (14.3)$$

Step 3: Calculate the transfer function ($H(z)$) of the DT filter (from the transfer function ($H_c(s)$) of the Butterworth CT filter) for $T_d = 1$ by using bilinear transformation concept.

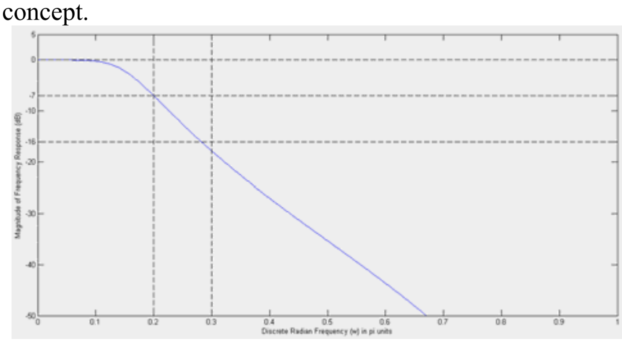

Figure 4 (a) The relationship between the magnitude in decibels (dB) of the frequency response of the digital filter, $20\log_{10}\left|H\left(e^{j\omega}\right)\right|$, and ω.

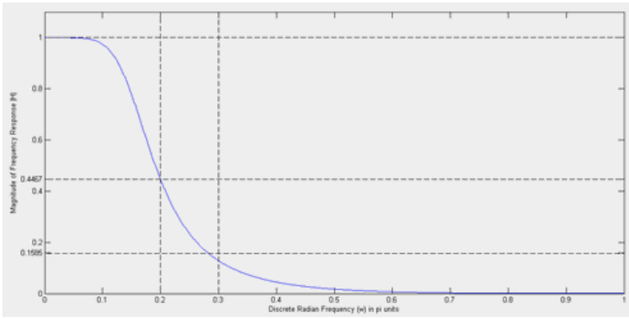

Figure 4 (b) The relationship between the magnitude of the frequency response of the digital filter, $\left|H\left(e^{j\omega}\right)\right|$, and digital frequency ω.

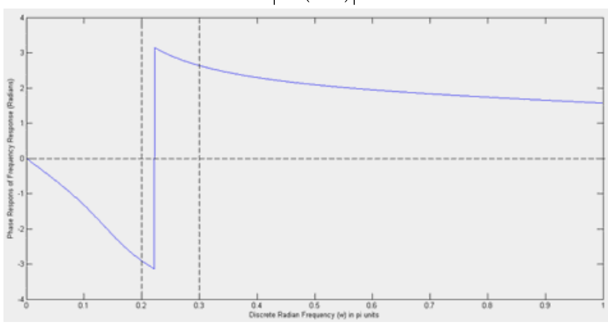

Figure 4 (c) The relationship between the phase of the frequency response of the digital filter, $\angle H\left(e^{j\omega}\right)$, and digital frequency ω.

$$H(z)=H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \tag{12}$$

$$H(z)=H_c\left(\frac{2}{(1)}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$$

$$H(z)=H_c\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$$

$$H(z)=\frac{0.1480}{\left[\begin{array}{l}\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^6+2.8100\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^5+3.9482\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^4\\+3.5168\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^3+2.0884\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2\\+0.7862\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)+0.1480\end{array}\right]} \tag{14.4}$$

First, the magnitude in decibels (dB), the magnitude and the phase of this frequency response of the analog filter $H_c(s)$ [17] can be illustrated as shown in figure 5, see next page after the references. Then, the magnitude and the phase of this frequency response of the digital filter $H(e^{j\omega})$ can be illustrated as in figure 6. From these experimental simulation results, the impulse invariance is perfectly transferred from the analog filter to the digital filter for magnitude perspective as shown in Fig. 5(a) and Fig. 6(a). Moreover, the phase of the frequency response of the digital filter, which is converted from analog filter, is different very slightly from the original analog filter as shown in Fig. 5(c) and Fig. 6(c).

V. DISCUSSION OF RESULTS AND CONCLUSION

We considered the analytical models of the digital IIR filter design approach based on Butterworth filtering and bilinear transformation for both statistical computation and simulation computation. The computer results illustrate the filter design procedure in both mathematical modelling and computer simulation. The simulation results of the analog filter and the digital filter were computationally analyzed to enhance the design quality. The computer results confirm this design technique has high accuracy for magnitude response and phase response requirements.

ACKNOWLEDGMENT

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RESULTS CONTINUE ON THE NEXT PAGE.

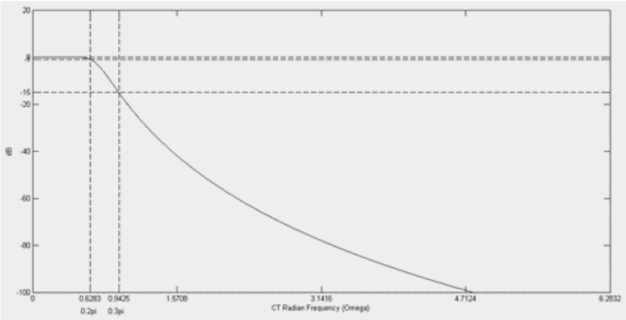

Figure 5 (a) The relationship between the magnitude in decibels (dB) of the frequency response of the analog filter, $20\log_{10}\left|H\left(e^{j\omega}\right)\right|$, and Ω.

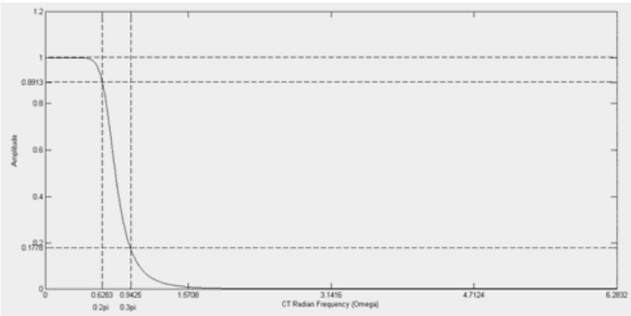

Figure 5 (b) The relationship between the magnitude of the frequency response of the analog filter, $\left|H_c\left(j\Omega\right)\right|$, and analog frequency Ω.

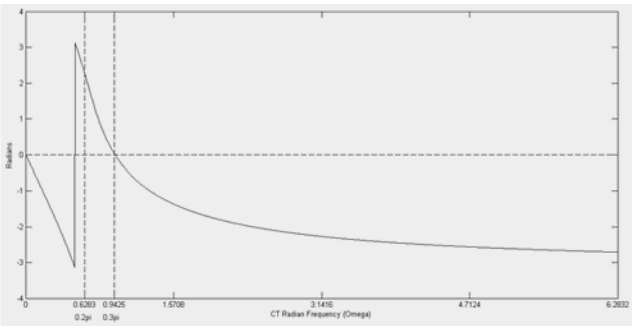

Figure 5 (c) The relationship between the phase of the frequency response of the analog filter, $\angle H_c\left(j\Omega\right)$, and analog frequency Ω.

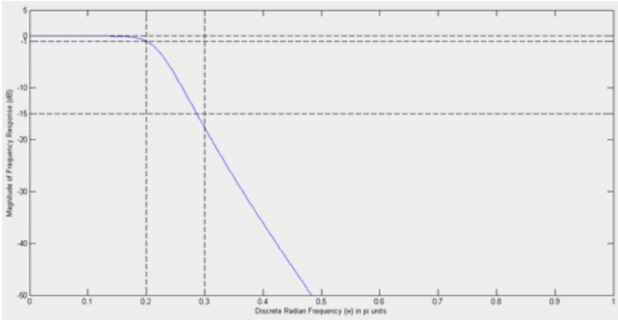

Figure 6 (a) The relationship between the magnitude in decibels (dB) of the frequency response of the digital filter, $20\log_{10}\left|H\left(e^{j\omega}\right)\right|$, and ω.

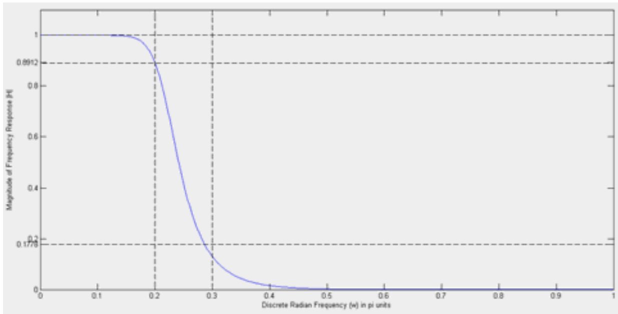

Figure 6 (b) The relationship between the magnitude of the frequency response of the digital filter, $\left|H\left(e^{j\omega}\right)\right|$, and digital frequency ω.

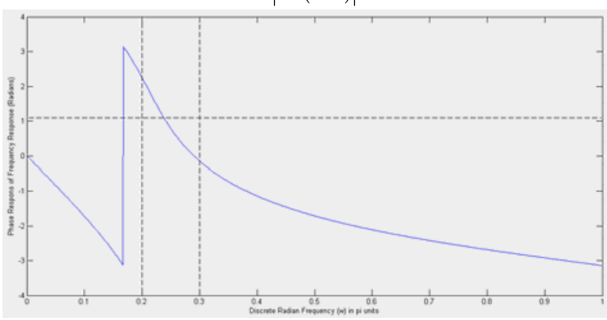

Figure 6 (c) The relationship between the phase of the frequency response of the digital filter, $\angle H\left(e^{j\omega}\right)$, and digital frequency ω.

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