Mathematical Modelling of the Spectral Response of Discrete-Time Linear Time-Invariant Systems for DSP and Digital Communications

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Abstract - Digital Signal Processing, DSP, and digital communication have dramatically developed to satisfy demand for their wide applications in mobile devices and related network technologies. The mathematical model of DT-LTI (Discrete-Time Linear Time-Invariant) systems has become a vital technique in modelling digital communication channels. The variables of magnitude response, phase response and group delay play major roles in this analysis to satisfy the complexity of digital communication channels and their implementation to improve performance. Our research project reported in this paper first developed the mathematical models of Discrete-Time (DT) linear time invariant (LTI) systems in magnitude response, phase response and group delay. This was followed by case studies to demonstrate their working principles in both mathematical formulations and graphical results to fully clarify and reinforce their understanding.

Keywords - Discrete-Time Fourier Transform (DT-FT), DT-LTI (Discrete-Time Linear Time-Invariance), Aliasing Problem, Digital Signal Processing (DSP).

I. INTRODUCTION

From the fact that the DSP and digital communication [1,5,10] , such as digital data [11], digital speech [6] and digital image [7, 8],have been dramatically developed for sanctifying the great demand, the mathematical model of DT-LTI (Discrete-Time Linear Time-Invariance) is one of the well-known mathematical model, especially for modeling digital communication channel such as 4G and 5G technology. For analyzing the communication performance, the modern digital communication channels, which usually are complicate, can be effectively modeled as Discrete-Time (DT) linear time invariant (LTI) for digital signals [2, 3, 4, 9, 12], the concept of magnitude response, phase response and group delay is very important for these propose.

II. INTRODUCTION OF DISCRETE-TIME LINEAR TIME-INVARIANCE SYSTEMS CONCEPT

By using both z-transform and DTFT (Discrete-Time Fourier transform), this section presents the description and analysis of DT-LTI (Discrete-Time Linear Time-Invariance) systems in more information and more perspective therefore the mathematical content in this section is an important foundation for LTI system implementation such as filter design.

From the mathematical characteristic of DT-LTI systems, the impulse response of the DT-LTI system, $h[n]$, in time domain can absolutely mathematical characterize the LTI system where the system output is $y[n]$ and the system input is $x[n]$. The mathematical relationship between the system output $y[n]$, the impulse response $h[n]$ and the system input $x[n]$ can be mathematical expresses by using convolution sum as following:

$$y[n] = x[n] * h[n] \;\rightarrow\; y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \tag{1}$$

By using z-transform, the mathematical relationship between the z-transform of the system input $X(z)$, the system function $H(z)$ (which is the z-transform of the impulse response of the LTI system) and the z-transform of the system output $Y(z)$ can be mathematical expresses by using convolution property as following:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \overset{Z}{\leftrightarrow} Y(z) = H(z)X(z) \tag{2}$$

III. MATHEMATICAL MODEL OF THE SPECTRAL RESPONSE OF DT LTI SYSTEMS

By using DTFT (Discrete-Time Fourier Transform), the frequency response of an DT-LTI system, $H(e^{j\omega})$, can be mathematical expresses as following:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \tag{3}$$

By using DTFT, the mathematical relationship between the Fourier transform of system input $X(e^{j\omega})$, the transfer

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function $H\left(e^{j\omega}\right)$ (that can be formulated by applying the Fourier transform to the impulse response of the DT-LTI system) and the Fourier transform of system output $Y\left(e^{j\omega}\right)$ can be mathematical expresses by using convolution property as following:

$$y[n]=\sum_{k=-\infty}^{+\infty} x[k]h[n-k] \overset{Z}{\leftrightarrow} Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X\left(e^{j\omega}\right) \quad (4)$$

A. Mathematical Model of The Spectral Response of Magnitude Response of DT-LTI Process Concept

Traditionally, the Fourier transform relationship between the system input $X\left(e^{j\omega}\right)$, the impulse response $H\left(e^{j\omega}\right)$ of the LTI system and the system output $Y\left(e^{j\omega}\right)$ are complex number and, thus, can be polar-form expressed in magnitude and phase as following:

$$\left|Y\left(e^{j\omega}\right)\right|=\left|H\left(e^{j\omega}\right)\right|\cdot\left|X\left(e^{j\omega}\right)\right| \quad (5.1)$$

where $\left|H\left(e^{j\omega}\right)\right|$ is defined as the magnitude response or the gain of the LTI system.

B. Mathematical Model of The Spectral Response of Phase Response of DT-LTI Process Concept

$$\angle Y\left(e^{j\omega}\right)=\angle H\left(e^{j\omega}\right)+\angle X\left(e^{j\omega}\right) \quad (5.2)$$

where $\angle H\left(e^{j\omega}\right)$ is defined as the phase response or the phase shift of the LTI system.

If the system input is undesirably adjusted in a deleterious appearance then the effects of an LTI system on a system signal (in Eq. (5.1) and Eq. (5.2)) is so called magnitude distortions and phase distortions, respectively. However, the phase angle of complex number cannot be defined uniquely because the complex number can be added with any integer multiple of 2π without impacting the complex number. Consequently, the principle value of the phase of system function $\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]$, so called wrapped phase, can be mathematically expressed as following:

$$-\pi<\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]\leq+\pi \quad (6)$$

The mathematical relationship between the principle value of the phase of system function $\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]$ and the phase of system function $\angle H\left(e^{j\omega}\right)$ can be mathematically expressed as following:

$$\angle H\left(e^{j\omega}\right)=\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]+2\pi r(\omega) \quad (7)$$

where $r(\omega)$ is a positive or negative integer

Traditionally, the principle value of the phase of system function $\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]$ will display discontinuities of 2π radians as shown in the following figure.

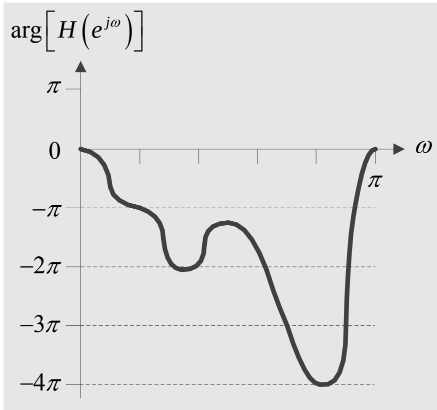

Figure 5.1 (a) Example of the phase of system function $\angle H\left(e^{j\omega}\right)$, which is a continuous characteristic.

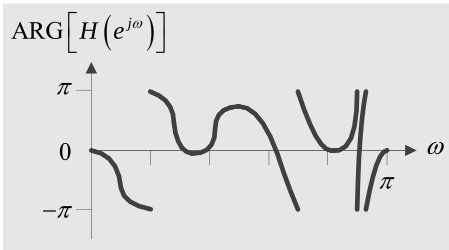

Figure 5.1 (b) Example of the principle value of the phase of system function $\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]$, which is a discontinuous characteristic.

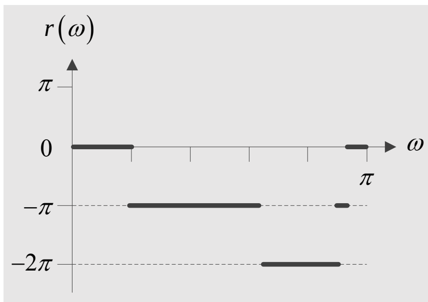

Figure 5.1 (c) Example of the integer $r(\omega)$.

The phase of system function, $\angle H\left(e^{j\omega}\right)$, can be illustrated in figure 5.1(a), which is a continuous characteristic but the principle value of the phase of system function, $\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]$, can be illustrated in figure 5.1(b), which is a discontinuous characteristic.

C. Mathematical Model of The Spectral Response of Group Delay of DT-LTI Process Concept

The group delay $\tau(\omega)$, which is another worldwide practical representation, can be mathematically defined as following:

$$\tau(\omega)=\mathrm{grd}\left[H\left(e^{j\omega}\right)\right] \rightarrow \tau(\omega)=-\frac{d}{d\omega}\left[\arg\left[H\left(e^{j\omega}\right)\right]\right] \quad (8)$$

The derivative of $\arg\left[H\left(e^{j\omega}\right)\right]$ is identical to the derivative of $\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]$ except for the discontinuities of $\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]$ because the derivative of $\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]$ at discontinuities are impulse functions.

IV. MATHEMATICAL MODEL OF CASES STUDY OF DT LTI SYSTEMS

D. Mathematical Model of Case I (The Frequency Response of the Ideal Delay DT-LTI System)

Determine the magnitude response $\left|H\left(e^{j\omega}\right)\right|$ and phase response $\angle H\left(e^{j\omega}\right)$ of the ideal delay DT-LTI system with the impulse response $h_{\mathrm{id}}[n]=\delta[n-n_d]$ (The ideal delay process is usually classified as one of the phase distortions).

The frequency response $H\left(e^{j\omega}\right)$ can be mathematically expressed as following:

$$h_{\mathrm{id}}[n]=\delta[n-n_d] \overset{F}{\leftrightarrow} H_{\mathrm{id}}\left(e^{j\omega}\right)=e^{-j\omega n_d} \quad (9)$$

The frequency response $H\left(e^{j\omega}\right)$ can be polar-form expressed in magnitude and phase as following

$$H_{\mathrm{id}}\left(e^{j\omega}\right)=e^{-j\omega n_d} \rightarrow \left|H_{\mathrm{id}}\left(e^{j\omega}\right)\right|=1 \quad (10.1)$$

$$\rightarrow \angle H_{\mathrm{id}}\left(e^{j\omega}\right)=-j\omega n_d, \quad \omega<\pi \quad (10.2)$$

E. Mathematical Model of Case II (The Frequency Response of the Ideal Lowpass DT-LTI system)

In general, the designing of ideal filters and LTI systems usually uses the linear-phase response instead of zero-phase response. The example of frequency response of the ideal lowpass filter with linear-phase response can be mathematically expressed as following:

$$H_{\mathrm{lp}}\left(e^{j\omega}\right)=\begin{cases} e^{-j\omega n_d} & |\omega|<\omega_c \\ 0 & \omega_c<|\omega|\le\pi \end{cases} \quad (11.1)$$

$$\overset{F}{\leftrightarrow} \quad h_{\mathrm{lp}}[n]=\frac{\sin\omega_c(n-n_d)}{\pi(n-n_d)}, \quad -\infty<n<\infty \quad (11.2)$$

The group delay response is a linear in phase as expressed:

$$\tau(\omega)=\mathrm{grd}\left[H\left(e^{j\omega}\right)\right]$$

$$\rightarrow \quad \tau(\omega)=-\frac{d}{d\omega}\left[\arg\left[H\left(e^{j\omega}\right)\right]\right]$$

$$\rightarrow \quad \tau(\omega)=-\frac{d}{d\omega}\left[-j\omega n_d\right]$$

$$\rightarrow \quad \tau(\omega)=jn_d \quad (11.3)$$

Typically, by using the principle of super position, a broadband signal can be expressed as a sum of many narrowband signals with different frequencies. Each narrowband signals will have different delay based on each group frequencies therefore the output signal is dispersion in time. (nonlinearity of phase or non-constant group delay result in time dispersion).

F. Mathematical Model of Case III (The Mathematical Influence of Group Delay and Attenuation)

This section aims to explain the influence of group delay and attenuation by analyzing the following DT LTI system with the system function as expresses as following equation.

$$H(z)=\left(\begin{array}{l}\left(\frac{\left(1-0.98e^{j0.8\pi}z^{-1}\right)\left(1-0.98e^{-j0.8\pi}z^{-1}\right)}{\left(1-0.8e^{j0.4\pi}z^{-1}\right)\left(1-0.8e^{-j0.4\pi}z^{-1}\right)}\right) \\ \times\prod_{k=1}^{4}\left(\frac{\left(c_k^*-z^{-1}\right)\left(c_k-z^{-1}\right)}{\left(1-c_kz^{-1}\right)\left(1-c_k^*z^{-1}\right)}\right)^2\end{array}\right) \quad (12)$$

$$\rightarrow \quad H(z)=H_1(z)H_2(z)$$

Where:

$$H_1(z)=\left(\frac{\left(1-0.98e^{j0.8\pi}z^{-1}\right)\left(1-0.98e^{-j0.8\pi}z^{-1}\right)}{\left(1-0.8e^{j0.4\pi}z^{-1}\right)\left(1-0.8e^{-j0.4\pi}z^{-1}\right)}\right)$$

$$H_2(z)=\prod_{k=1}^{4}\left(\frac{\left(c_k^*-z^{-1}\right)\left(c_k-z^{-1}\right)}{\left(1-c_kz^{-1}\right)\left(1-c_k^*z^{-1}\right)}\right)^2$$

with $c_k=0.95e^{j(0.15\pi+0.02\pi k)}$ for $k=1,2,3,4$

This system function has pole-zero plot as shown in the following figure. The system function $H_1(z)$ in the above equation consists of two complex conjugate poles (at $z=0.8e^{\pm j0.4\pi}$) and two complex conjugate zeros (at $z=0.98e^{\pm j0.8\pi}$) as shown in figure 5.2. Next, the system function $H_2(z)$ in the above equation consists of two complex conjugate poles (at $z=c_k=0.95e^{\pm j(0.15\pi+0.02\pi k)}$ for $k=1,2,3,4$) with double order poles and two complex conjugate zeros (at $z=1/c_k=1/0.95e^{\mp j(0.15\pi+0.02\pi k)}$ for $k=1,2,3,4$) with double order zeros as shown in figure 5.2. From filter characterized analysis, the $H_2(z)$ is an all-pass system (which is discussed later) because $|H_2(z)|=1,\forall\omega$ and, moreover, the $H_2(z)$ presents a group delay for a narrow band of frequencies.

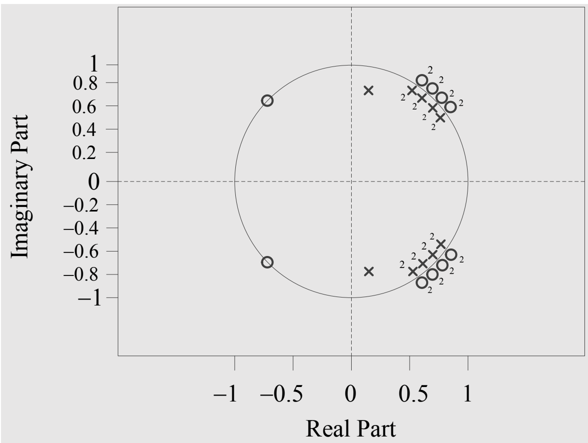

Figure 5.2 The pole-zero plot for the DT-LTI system example

The phase response (or unwrapped phase response) of this system function $\angle H(z)$ and the principle value phase response (or wrapped phase response) of this system function $\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]$ can be expressed in figure 5.3(a) and 5.3(b), respectively. From this figure, the principle value phase response has multiple discontinuities because of modulo 2π calculation.

From figure 5.3(b), the phase response (or unwrapped phase response) of this system function $\angle H(z)$ is monotonically decreasing excluding about $\omega=\pm0.8\pi$ therefore the group delay $\tau(\omega)$ is positive throughout excluding about $\omega=\pm0.8\pi$. Moreover, the group delay $\tau(\omega)$ is large positive in the frequency region between $0.17\pi<|\omega|<0.23\pi$ (or the unwrapped phase response of this system function $\angle H(z)$ is large negative slope).

Later, the output signal is mathematically examined if the DT input signal $x[n]$ is comprised of three narrowband pulses, which is time-domain apportioned, as shown in figure 5.5(a) and figure 5.5(b) for time-domain and frequency domain, respectively. This input signal $x[n]$ can be mathematically expressed as following equation.

$$x[n]=x_3[n]+x_1[n-M-1]+x_2[n-2M-2] \text{ for } M=60 \tag{13.1}$$

where

$$x_1[n]=w[n]\cos(0.2\pi n) \tag{13.2}$$

$$x_2[n]=w[n]\cos(0.4\pi n-\pi/2) \tag{13.3}$$

$$x_3[n]=w[n]\cos(0.8\pi n+\pi/5) \tag{13.4}$$

$$w[n]=\begin{cases}0.54-0.46\cos(2\pi n/M) & ,0\le n\le M=60\\ 0 & ,\text{otherwise}\end{cases} \tag{13.5}$$

The DT input signal $x[n]$ is comprised of the highest frequency pulse $x_3[n]$, the lowest frequency pulse $x_1[n]$ and, followed by the medium frequency pulse $x_2[n]$, as shown in figure 5.5(a). The frequency of three sinusoidal components are $\omega_1=0.2\pi$, $\omega_2=0.4\pi$ and $\omega_3=0.8\pi$ therefore the peak of the signal spectrum, as shown in figure 5.5(b), is located at are $\omega_1=0.2\pi$, $\omega_2=0.4\pi$ and $\omega_3=0.8\pi$, respectively.

Consequently, if this DT input signal $x[n]$ is processed by this system function $H(z)$ then each frequency components of this DT input signal $x[n]$ is processed with different magnitude response (as shown in figure 5.6) as follow:

- For the lowest frequency pulse $x_1[n]=w[n]\cos(0.2\pi n)$ which is spectrum cluster focus at $\omega=\omega_1=0.2\pi$, the output for this spectrum component is little enlarged in magnitude.
- For the medium frequency pulse $x_2[n]=w[n]\cos(0.4\pi n-\pi/2)$ which is spectrum cluster

focus at $\omega = \omega_2 = 0.4\pi$, the output for this spectrum component is double enlarged in magnitude.

- For the highest frequency pulse $x_3[n] = w[n]\cos(0.8\pi n + \pi/5)$ which is spectrum cluster focus at $\omega = \omega_3 = 0.8\pi$, the output for this spectrum component is almost completely suppressed to zero in magnitude.

Next, for the group delay perspective, each frequency components of this DT input signal $x[n]$ is differently processed (as shown in figure 5.6) as following.

- For the lowest frequency pulse $x_1[n] = w[n]\cos(0.2\pi n)$ which is spectrum cluster focus at $\omega = \omega_1 = 0.2\pi$, the output for this spectrum component is more severely delayed (≈ 150 samples) than other spectrum components.
- For the medium frequency pulse $x_2[n] = w[n]\cos(0.4\pi n - \pi/2)$ which is spectrum cluster focus at $\omega = \omega_2 = 0.4\pi$, the output for this spectrum component is little delayed (≈ 10 samples).

For the analysis result of this section, the process of wideband signals is more complicated than this analyzed case because each spectrum component of the wideband signal is differently processed in both magnitude and phase perspective.

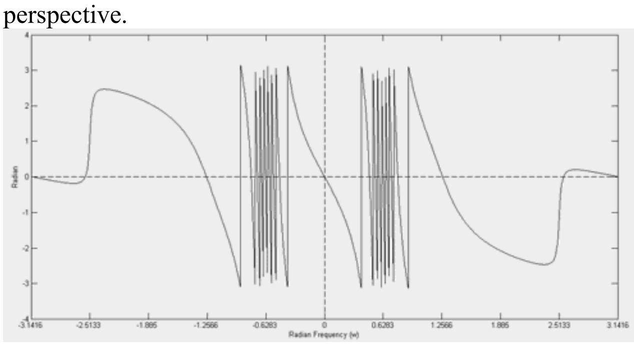

Figure 5.3 (a) The principle value phase response (or wrapped phase response) of this system function $\text{ARG}\left[H\left(e^{j\omega}\right)\right]$

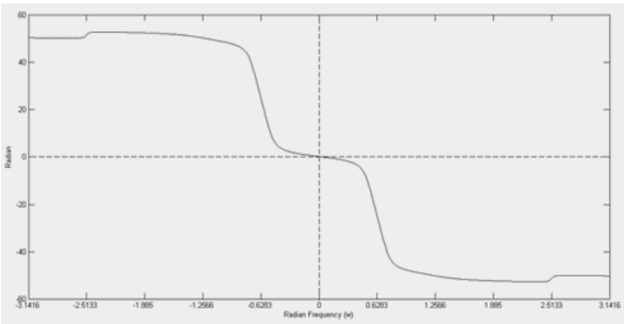

Figure 5.3 (b) The phase response (or unwrapped phase response) of this system function $\angle H(z)$

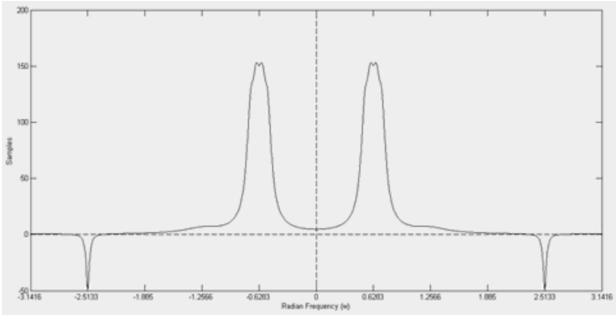

Figure 5.4 (a) The group delay of this system function ($\tau(\omega) = \text{grd}\left[H\left(e^{j\omega}\right)\right]$)

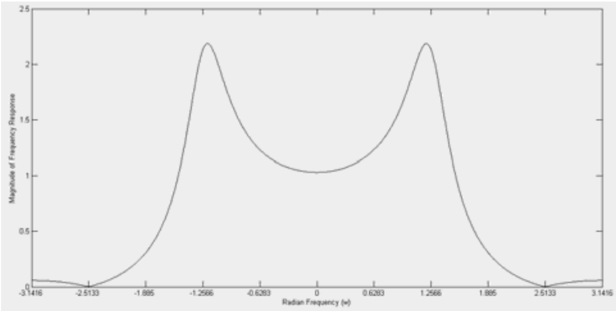

Figure 5.4 (b) The magnitude response of this system function ($\left|H\left(e^{j\omega}\right)\right|$)

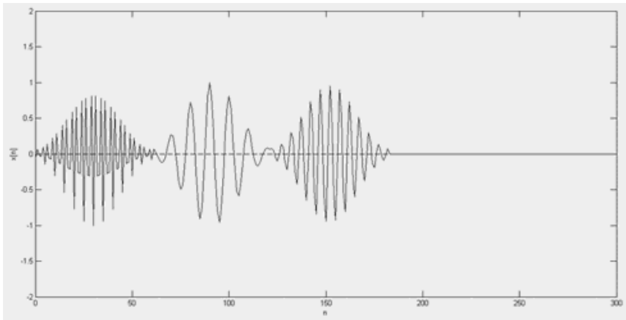

Figure 5.5 (a) The input signal $x[n]$ (in time domain)

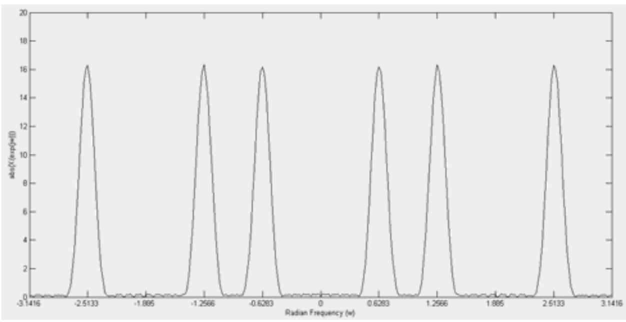

Figure 5.5 (b) The spectrum magnitude of input signal $\left|H\left(e^{j\omega}\right)\right|$ (in frequency domain)

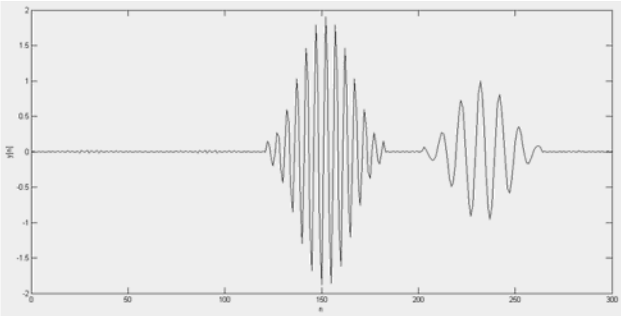

Figure 5.6 The output signal which is processed from the system function $H(z)$ by the input $x[n]$

-V. CONCLUSION

The primary propose of this project is the development of mathematical models for Discrete-Time (DT) linear time invariant (LTI) systems and their systematic application to meet demand in the DSP [2, 4] and digital communication [11] areas. This will provide solid foundations for further modelling, especially for new digital communication channels such as 4G and 5G technology. A large set of mathematical expressions and graphical results were given to illustrate the work. Many studied cases were provided and analyzed in terms of magnitude response, phase response and group delay perspective.

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