Mathematical Models of Multi-Rate Signal Processing Concepts for Down/Up Sampling in Multi-State and Poly-Phase Decompositions for DSP and Digital Communication Applications

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***Abstract* - Nyquist-Shannon sampling principles extract Discrete-Time (DT) signals from Continuous-Time (CT) signals and are crucial in all Digital Signal Processing (DSP) areas, both in purely Discrete-Time (DT) systems and in processed signals as DT signals or CT analog signals. Multi-rate signal processing makes use of down-sampling, up-sampling, compression and expansion in many applications. Our work in this paper aims to highlight the mathematical models of multi-rate signal processing concepts on down/up sampling, multi-state and poly-phase decompositions for DSP and digital communications. We formulate and graphically illustrate the mathematical models in a case study of the analysis of multi-rate signal processing.**

Keywords - Continuous-Time Fourier Transform (CT-FT), Discrete-Time Fourier Transform (DT-FT), Multi-rate Signal Processing, Aliasing Problem, Digital Signal Processing (DSP).

I. INTRODUCTION OF MATHEMATICAL MODEL OF MULTI-RATE SIGNAL PROCESSING CONCEPT

From the great insistence of both digital systems (so called discrete-time systems) and digital signals (so called discrete-time signals [1, 5, 13]) in many advance digital applications such as digital speech [2, 4], digital image [6, 7,8] and digital communication data [14], the digital systems and digital signals are fundamental advance electronic elements based on microcontrollers / microprocessors (such as smart phone, MP3 player, digital camera and PDA, etc.). The practical recorded signals, which normally are Continuous-Time (CT) signals, are mathematically changed to be Discrete-Time (DT) signals by Nyquist-Shannon sampling principle [9,10,11,12] that is the systematic sampling founded on different parametric basis distribution modeling [3,15] for applying on both aliasing and non-aliasing for periodic signals [9].

Traditionally, the multi-rate signal processing relates to make use of down-sampling, up-sampling, compressors and expanders in a several ways to raise the performance of digital signal processing or DSP systems such as if the new sampling period of the system is changed to be $1.03T$ then the system is first up-sampling/interpolating by $L = 100$ (with the low-pass filter, which has the cutoff frequency at $\omega_c = \pi/103$) and the systems is later down-sampling by $M = 103$ from the sampling rate changing concept. However, from the classical straightforward designing of a filter concept, the new system with dramatically high sampling rate (by $L = 100$) has the high complexity and high computation for generating each output signals. Alternatively, another designing of a filter concept, so called multi-rate signal processing, can reduce the high complexity and high computation of the new system. Due to the large implementation of the multi-rate signal processing, this article is mainly presented only on the interchanging concept of the filter process and down-sampling/up-sampling process, and the poly-phaser decomposition concept.

II. MATHEMATICAL MODELS OF MULTI-RATE SIGNAL PROCESSING CONCEPT

A. Mathematical Analysis of The Interchanging Concept of The Filter Process and Down-sampling/Up-sampling Process

In general, the filter process and down-sampling process can be interchanged as shown in the following figure

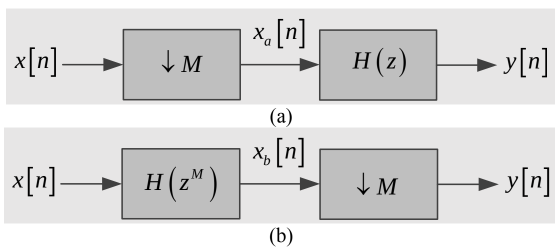

Figure 1. The block diagram of the interchanging concept of the filter process and down-sampling process.

By using Discrete-Time analysis with Fourier analysis, the down-sampling mathematical relationship between a discrete-time (DT) input signal $x[n]$ and a discrete-time (DT) filtered signal $x_b[n]$ can be mathematically defined as:

$$X_b\left(e^{j\omega}\right) = H\left(e^{j\omega M}\right) X\left(e^{j\omega}\right) \tag{1}$$

From Discrete-Time analysis with Fourier analysis, the mathematical relationship between a discrete-time (DT) input signal $x[n]$ and a discrete-time (DT) down-sampled signal $y[n]$ can be mathematically defined as:

$$Y\left(e^{j\omega}\right)=\frac{1}{M}\sum_{i=0}^{M-1}X_d\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right) \quad (2)$$

By substituting $X_b\left(e^{j\omega}\right)=H\left(e^{j\omega M}\right)X\left(e^{j\omega}\right)$ of equation (1) into equation (2), the Fourier transform of a discrete-time (DT) down-sampled signal $Y\left(e^{j\omega}\right)$ can be mathematically defined as:

$$Y\left(e^{j\omega}\right)=\frac{1}{M}\sum_{i=0}^{M-1}X\left(e^{j(\omega/M-2\pi i/M)}\right)H\left(e^{j(\omega-2\pi i)}\right)$$

$$\rightarrow\ Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)\frac{1}{M}\sum_{i=0}^{M-1}X\left(e^{j(\omega/M-2\pi i/M)}\right) \quad (3)$$

Where:

$$H\left(e^{j(\omega-2\pi i)}\right)=H\left(e^{j\omega}\right)\rightarrow Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X_a\left(e^{j\omega}\right) \quad (4)$$

The term $Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X_a\left(e^{j\omega}\right)$ is accords to figure 1(a) thereby the figure 1(a) and figure 1(b) are entirely identical.

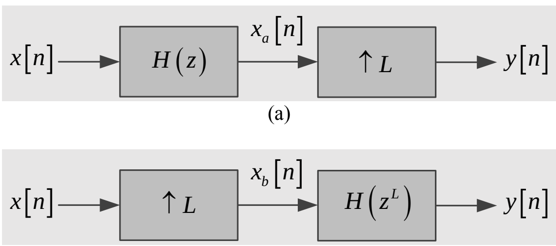

Figure 2. The block diagram of the interchanging concept of the filter process and up-sampling process.

Like the mathematical analysis of down-sampling process in the preceding section, the up-sampling mathematical relationship between a discrete-time (DT) input signal $y[n]$ and a discrete-time (DT) filtered signal $x_a[n]$ can be mathematically defined as:

$$Y\left(e^{j\omega}\right)=X_a\left(e^{j\omega L}\right)\qquad\rightarrow\ Y\left(e^{j\omega}\right)=X\left(e^{j\omega L}\right)H\left(e^{j\omega L}\right) \quad (5)$$

From Discrete-Time analysis with Fourier analysis, the mathematical relationship between a discrete-time (DT) signal $x_b[n]$ and a discrete-time (DT) input signal $x[n]$ can be mathematically defined as:

$$X_b\left(e^{j\omega}\right)=X\left(e^{j\omega L}\right) \quad (6)$$

By substituting $X_b\left(e^{j\omega}\right)=X\left(e^{j\omega L}\right)$ of equation (4.56) into equation (4.55), the Fourier transform of a discrete-time (DT) up-sampled signal $Y\left(e^{j\omega}\right)$ can be mathematically defined as:

$$Y\left(e^{j\omega}\right)=X\left(e^{j\omega L}\right)H\left(e^{j\omega L}\right)\rightarrow\ Y\left(e^{j\omega}\right)=X_b\left(e^{j\omega}\right)H\left(e^{j\omega L}\right)$$
$$\rightarrow Y\left(e^{j\omega}\right)=H\left(e^{j\omega L}\right)X_b\left(e^{j\omega}\right) \quad (7)$$

The term $Y\left(e^{j\omega}\right)=H\left(e^{j\omega L}\right)X_b\left(e^{j\omega}\right)$ is accords to figure 2(b) thereby the figure 2(a) and figure 2(b) are entirely identical.

B. Mathematical Analysis of the Multistate of Down-Sampling/Up-Sampling Process

It is mandatory to apply a very long impulse response filter for obtaining satisfied approximation to the appropriate low-pass filter where the ratio of down-sampling or up-sampling is large. Consequently, the applying of multistate down-sampling or up-sampling can dramatically reduce the computational complexity of the very long impulse response filter.

The two-state down-sampling system is shown in figure 3(a) where the overall ration of down-sampling is $M=M_1M_2$. The two low-pass filters are required in this system: the low-pass filter $H_1(z)$ with cutoff frequency π/M_1 and the low-pass filter $H_2(z)$ with cutoff frequency π/M_2 therefore the satisfied cutoff frequency of the overall single state down-sampling system would be $\pi/M=\pi/(M_1M_2)$, which is dramatically smaller than the cutoff frequency of either of the two filters (π/M_1 or π/M_2). Due to this reason, the implementation of the two-state concept is usually greatly more powerful than a single-state process.

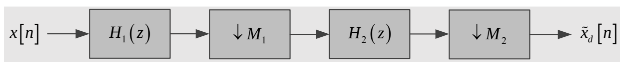

Figure 3(a). The block diagram of the two state down-sampling concept

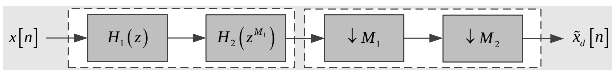

Figure 3(b). The altered block diagram of the two state down-sampling concept by applying the interchanging concept of the filter process and down-sampling process

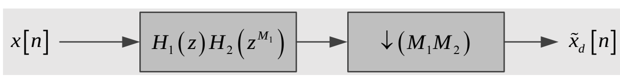

Figure 3(c). The identified block diagram of the one state down-sampling concept for two state down-sampling process

From the above figure, the transfer function of the identified low pass filter of the one-state process can be mathematically defined as:

$$H(z) = H_1(z) H_2\left(z^{M_1}\right) \tag{8}$$

From the above equation, the filter with the above transfer function (in equation (8)) is referred as an interpolated FIR filter and its impulse response can be mathematically defined as:

$$h[n] = h_1[n] * \sum_{k=-\infty}^{+\infty} h_2[k] \delta[n - kM_1] \tag{9}$$

By applying the identical concept, the multistate concept of up-sampling process can be applied on the mathematical relationship between one-state process and two-state process as shown in following figures

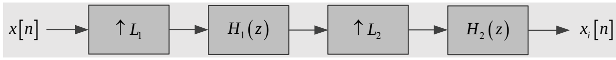

Figure 4(a). The block diagram of the two state up-sampling concept.

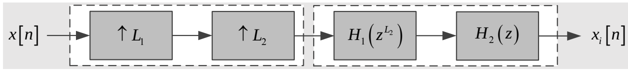

Figure 4(b). The altered block diagram of the two state up-sampling concept by applying the interchanging concept of the filter process and up-sampling process.

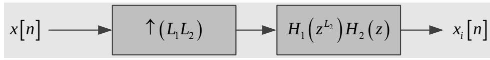

Figure 4(c). The identified block diagram of the one state up-sampling concept for two state up-sampling process.

C. *Mathematical Analysis of the Poly-Phase Decompositions*

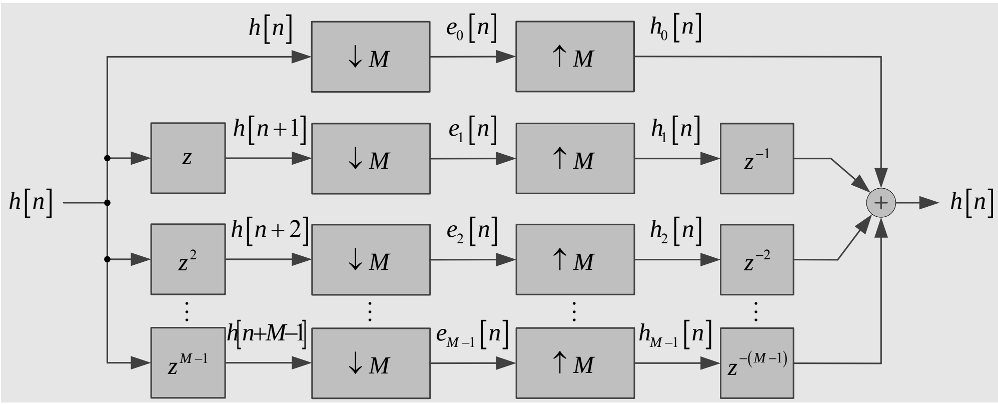

Figure 5. The polyphaser decomposition of filter $h[n]$ based on $e_k[n]$

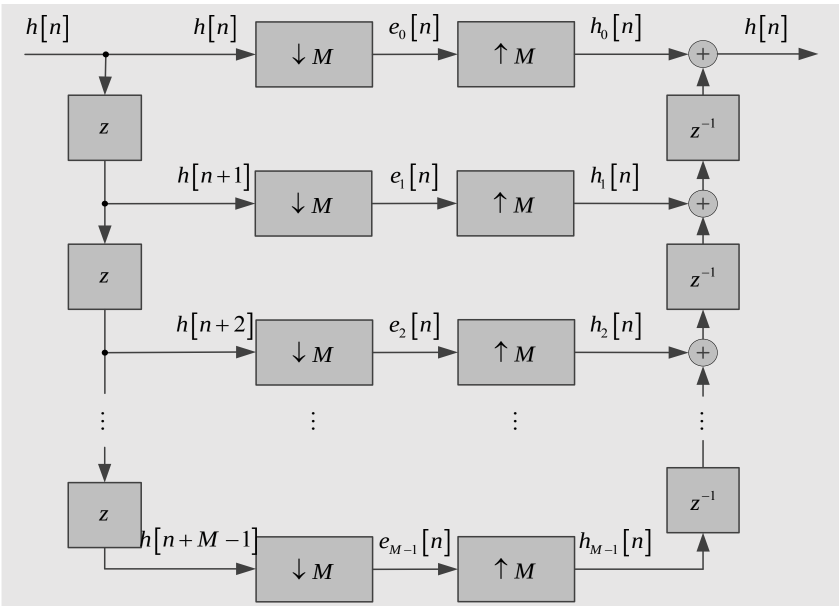

Figure 6. The polyphaser decomposition of filter $h[n]$ based on $e_k[n]$ by applying chain delays.

The poly-phaser decomposition processing is relates to the mathematical representation of the sequence in the form of M subsequences (each subsequence is compounded by M^{th} value of successively delayed of the original sequence) by using the principle of superposition. Consequently, the poly-phaser decomposition processing is usually applied on the filter impulse response because this process will mathematically convert to the linear filter context.

The impulse response $h[n]$ can be mathematically decomposed into M subsequences $h_k[n]$ for $k = 0,1,\ldots,(M-1)$ as following:

$$h_k[n] = \begin{cases} h[n-k] & n = \text{integer multiple of } M \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

The original impulse response $h[n]$ can be reconstructed by summing all successively delayed subsequences as following:

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k] \quad (11)$$

The sequence $e_k[n]$ shown in figure 5 and figure 6, which are referred as poly-phaser components of $h[n]$, can be mathematically defined as

$$e_k[n] = h[nM+k] = h_k[nM] \quad (12)$$

By using Z-transform, the $H(z)$ can be poly-phase decomposed into a sum of delayed poly-phaser component filters as following equation and its filter structure can be shown in figure 7.

$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k} \quad (13)$$

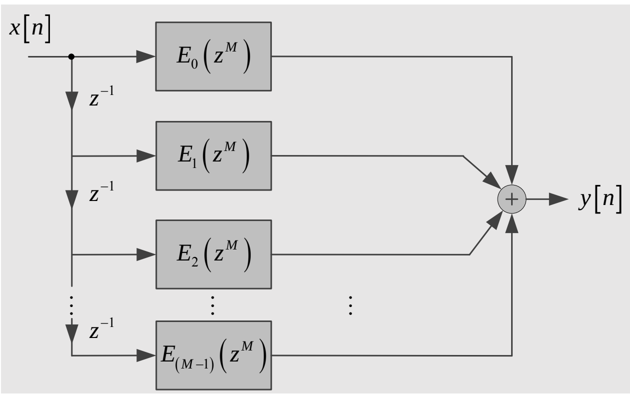

Figure 7. The poly-phaser decomposition structure of $h[n]$

D. Mathematical Analysis of The Poly-phase Decompositions for Down-sampling Filter

The implementation of poly-phase decompositions for down-sampling filter (as shown in figure 4.38) is one of the most crucial applications

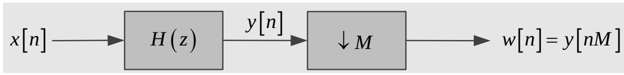

Figure 8. The down-sampling process

In order to reduce the computational complexity, the thrown away sampling (in the second block diagram as shown in figure 8) should not be computed. The $H(z)$ can be can be mathematically decomposed as following:

$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k} \quad \text{where } e_k[n] = h[nM+k]$$

The down-sampling process in figure 8 can be restructured to be the poly-phase decomposition processing in figure 9 and the poly-phase decomposition structure can be simplified by using the filter interchange property (as discussed in the mathematical analysis of the interchanging concept of the filter process and down-sampling/up-sampling process in figure 1) as figure 10.

The computational advantage of the poly-phase decomposition structure (in figure 10) can be expressed in this section. Assume that the input signal $x[n]$ is clocked at a rate one sample per unit time and assume that the filter $H(z)$ is an N-point FIR filter. The original down sampling process (in figure 8) requires M multiplication and (N-1) additions per unit time but the poly-phase decomposition structure (in figure 10) requires only N/M multiplication and (N/M-1)+(M-1) additions per unit time.

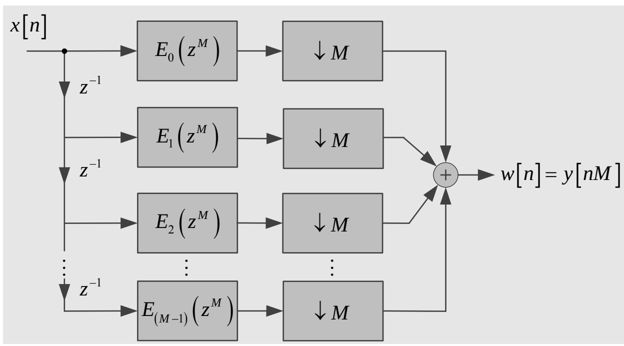

Figure 9. The poly-phaser decomposition process of the down-sampling process

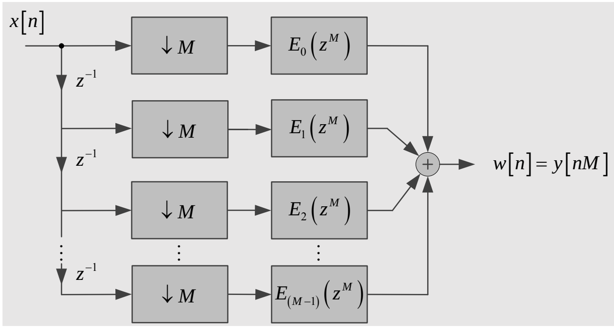

Figure 10. The poly-phaser decomposition process of the down-sampling process using interchange property.

E. Mathematical Analysis of The Poly-phase Decompositions for Up-sampling Filter

The implementation of poly-phase decompositions for down-sampling filter (as shown in figure 11) is one of the most crucial applications.

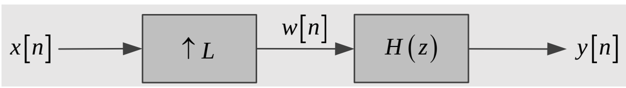

Figure 11. The up-sampling process

In order to reduce the computational complexity, the thrown away sampling (in the second block diagram as shown in figure 11) should not be computed.

The up-sampling process in figure 11 can be restructured to be the poly-phase decomposition processing in figure 12 and the poly-phase decomposition structure can be simplified by using the filter interchange property (as discussed in the mathematical analysis of the interchanging concept of the filter process and down-sampling/up-sampling process in figure 12) as figure 13.

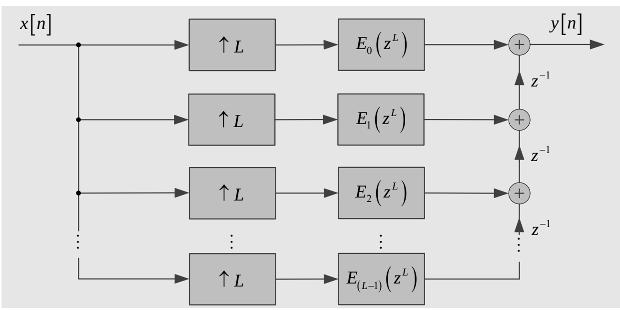

Figure 12. The poly-phaser decomposition process of the up-sampling process

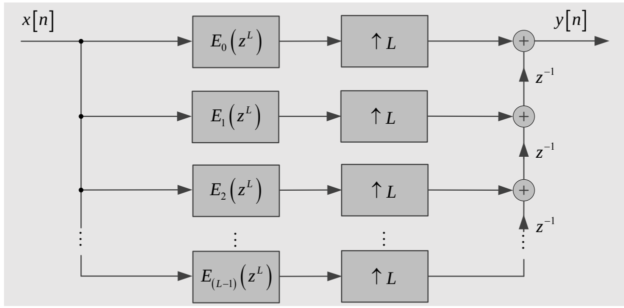

Figure 13. The poly-phaser decomposition process of the up-sampling process using interchange property

The computational advantage of the poly-phase decomposition structure (in figure 13) can be expressed in this section. Assume that the input signal $x[n]$ is clocked at a rate one sample per unit time then $u[n]$ is clocked at a rate of L samples per unit time and assume that the filter $H(z)$ is an N-point FIR filter. The original down sampling process (in figure 11) requires NL multiplication and (NL-1) additions per unit time but the poly-phase decomposition structure (in figure 13) requires only $L(N/L)$ multiplication and $L(N/L\text{-}1)$ additions per unit time for the poly-phase filter set and (L-1) additions to obtain $y[n]$.

The computational advantage on both down-sampling and up-sampling process is based on the poly-phaser decomposition reorganization therefore the filtering on both down-sampling and up-sampling process operates at lower sampling rate than the original operation.

F. Mathematical Analysis of the Multi-Rate Filter Banks

One of the most widely crucial applications based on poly-phaser structure for down-sampling and up-sampling process is the filter bank structure of audio and speech signal processing such as a two-channel analysis and synthesis filter bank processing for audio and speech signal processing where the main objective of the analysis processing is for separating the input spectrum bandwidth $x[n]$ into a low-pass band $v_0[n]$ and high-pass band $v_1[n]$ as shown in the following figure.

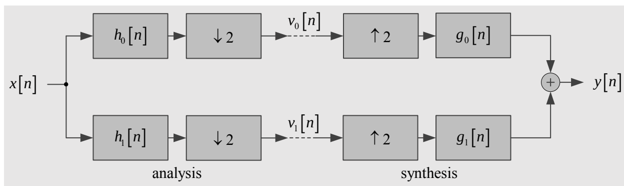

Figure 14. The filter bank of the two-channel analysis and synthesis

In the above analysis/synthesize process, the impulse response of low-pass and high-pass filter is $h_0[n]$ and $h_1[n]$, respectively where $h_1[n]=e^{j\pi n}h_0[n]$ or

$H_1\left(e^{j\omega}\right)=H_0\left(e^{j(\omega-\pi)}\right)$. If the low-pass filter $H_0\left(e^{j\omega}\right)$ has nominal passband $0\le|\omega|\le\pi/2$ then the high-pass filter $H_1\left(e^{j\omega}\right)$ has nominal passband $\pi/2\le|\omega|\le\pi$.

By using Discrete-Time analysis with Fourier analysis, the mathematical relationship between a discrete-time (DT) input signal $x[n]$ and a discrete-time (DT) output signal $y[n]$ can be mathematically defined as:

$$Y\left(e^{j\omega}\right)=\begin{cases}\frac{1}{2}\left[G_0\left(e^{j\omega}\right)H_0\left(e^{j\omega}\right)+G_1\left(e^{j\omega}\right)H_1\left(e^{j\omega}\right)\right]X\left(e^{j\omega}\right)\\+\frac{1}{2}\left[G_0\left(e^{j\omega}\right)H_0\left(e^{j(\omega-\pi)}\right)+G_1\left(e^{j\omega}\right)H_1\left(e^{j(\omega-\pi)}\right)\right]X\left(e^{j(\omega-\pi)}\right)\end{cases} \tag{14}$$

If the analysis and synthesis filter are ideal filters therefore the input spectrum bandwidth ($0\le|\omega|\le\pi$) is perfectly separated into $0\le|\omega|\le\pi/2$ and $\pi/2\le|\omega|\le\pi$ without overlapping then it can be concluded that $Y\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)$.

From the above equation, the second term ($\frac{1}{2}\left[G_0\left(e^{j\omega}\right)H_0\left(e^{j(\omega-\pi)}\right)+G_1\left(e^{j\omega}\right)H_1\left(e^{j(\omega-\pi)}\right)\right]X\left(e^{j(\omega-\pi)}\right)$), which refers to the important aliasing distortion that is caused from down-sampling process, can be excluded by the filter with the following passband.

$$G_0\left(e^{j\omega}\right)H_0\left(e^{j(\omega-\pi)}\right)+G_1\left(e^{j\omega}\right)H_1\left(e^{j(\omega-\pi)}\right)=0 \tag{15}$$

This condition of this filter is so called the alias cancellation condition, which can be mathematical express as following.

$$h_1[n]=e^{j\pi n}h_0[n] \quad\Leftrightarrow\quad H_1\left(e^{j\omega}\right)=H_0\left(e^{j(\omega-\pi)}\right) \tag{16.1}$$

$$g_0[n]=2h_0[n] \quad\Leftrightarrow\quad G_0\left(e^{j\omega}\right)=2H_0\left(e^{j\omega}\right) \tag{16.2}$$

$$g_1[n]=-2h_1[n] \quad\Leftrightarrow\quad G_1\left(e^{j\omega}\right)=-2H_1\left(e^{j(\omega-\pi)}\right) \tag{16.3}$$

The filter $h_0[n]$ and $h_1[n]$ is so called the quadrature mirror filter because the equation (16.1) enforces mirror symmetry about $\omega=\pi/2$.

By substituting $G_0\left(e^{j\omega}\right)=2H_0\left(e^{j\omega}\right)$ from equation (16.2) in $Y\left(e^{j\omega}\right)$ from equation (14), the output signal $y[n]$ can be mathematically defined as

$$Y\left(e^{j\omega}\right)=\begin{cases}\frac{1}{2}\left[G_0\left(e^{j\omega}\right)H_0\left(e^{j\omega}\right)+G_1\left(e^{j\omega}\right)H_1\left(e^{j\omega}\right)\right]X\left(e^{j\omega}\right)\\+\frac{1}{2}\left[G_0\left(e^{j\omega}\right)H_0\left(e^{j(\omega-\pi)}\right)+G_1\left(e^{j\omega}\right)H_1\left(e^{j(\omega-\pi)}\right)\right]X\left(e^{j(\omega-\pi)}\right)\end{cases}$$

$$\rightarrow\; Y\left(e^{j\omega}\right)=\begin{cases}\frac{1}{2}\left[2H_0\left(e^{j\omega}\right)H_0\left(e^{j\omega}\right)-2H_1\left(e^{j(\omega-\pi)}\right)H_1\left(e^{j\omega}\right)\right]X\left(e^{j\omega}\right)\\+\frac{1}{2}\left[2H_0\left(e^{j\omega}\right)H_0\left(e^{j(\omega-\pi)}\right)-2H_1\left(e^{j(\omega-\pi)}\right)H_1\left(e^{j(\omega-\pi)}\right)\right]X\left(e^{j(\omega-\pi)}\right)\end{cases}$$

$$\rightarrow\; Y\left(e^{j\omega}\right)=\begin{cases}\left[H_0\left(e^{j\omega}\right)H_0\left(e^{j\omega}\right)-H_1\left(e^{j(\omega-\pi)}\right)H_1\left(e^{j\omega}\right)\right]X\left(e^{j\omega}\right)\\+\left[H_0\left(e^{j\omega}\right)H_0\left(e^{j(\omega-\pi)}\right)-H_1\left(e^{j(\omega-\pi)}\right)H_1\left(e^{j(\omega-\pi)}\right)\right]X\left(e^{j(\omega-\pi)}\right)\end{cases}$$

$$\rightarrow\; Y\left(e^{j\omega}\right)=\left[H_0^2\left(e^{j\omega}\right)-H_1^2\left(e^{j(\omega-\pi)}\right)\right]X\left(e^{j\omega}\right) \tag{17}$$

If there is perfect reconstruction with possible *M* sample delay then $Y\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)$ and $H_0^2\left(e^{j\omega}\right)-H_1^2\left(e^{j(\omega-\pi)}\right)=e^{-j\omega M}$ or $h_0[n]=c_0\delta[n-2n_0]+c_1\delta[n-2n_1-1]$ where n_0 and n_1 are arbitrarily integers and $c_0c_1=\frac{1}{4}$.

If the filter is enforced by equation (16.1) to (16.3) then the filter used in the analysis and synthesis processing (figure 14) can be FIR or IIR filter.

In order to reduce the computational complexity, a two-channel analysis filter bank processing can be reconstructed by using the poly-phaser decomposition as figure 15(a) and following equations:

$$e_{00}[n]=h_0[2n] \tag{18.1}$$

$$e_{01}[n]=h_0[2n+1] \tag{18.2}$$

$$e_{10}[n]=h_1[2n]=e^{j2\pi n}h_0[2n]=e_{00}[n] \tag{18.3}$$

$$e_{11}[n]=h_1[2n+1]=e^{j2\pi n}e^{j\pi}h_0[2n+1]=-e_{01}[n] \tag{18.4}$$

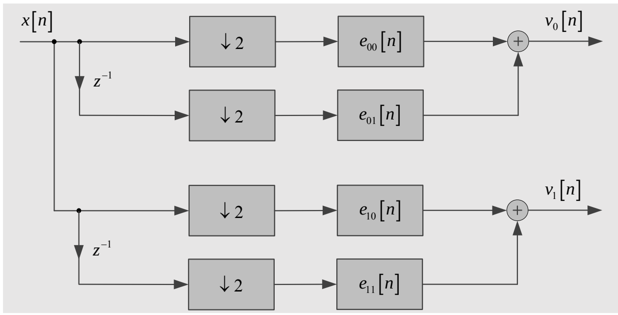

Figure 15(a) The poly-phaser decomposition process of filter bank of the two-channel analysis

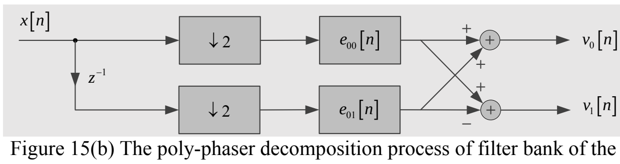

Figure 15(b) The poly-phaser decomposition process of filter bank of the two-channel analysis

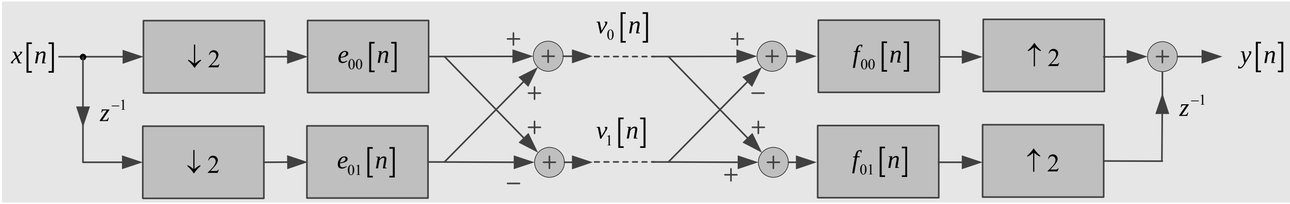

Figure 16. The polyphaser decomposition process of filter bank of the two-channel analysis and synthesis

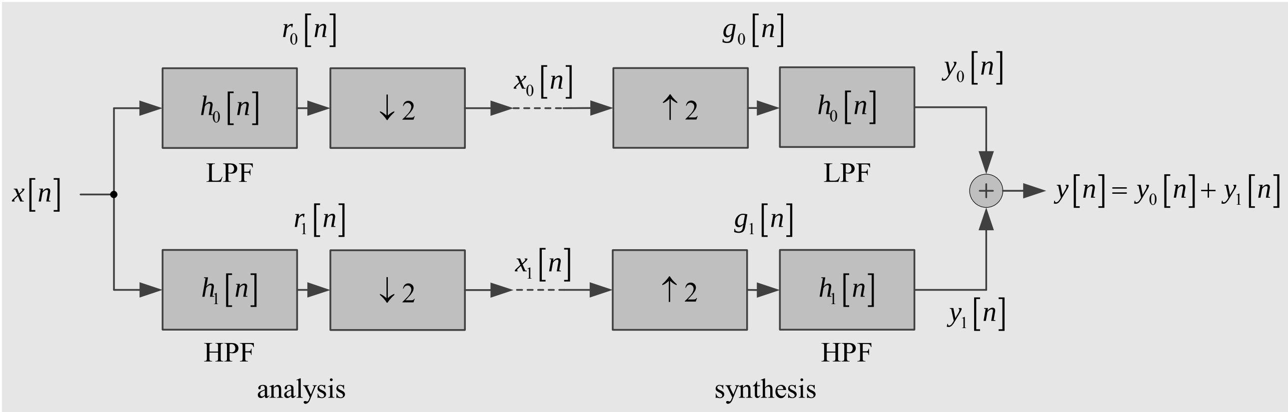

Figure 17(a). The LTI system of Mathmatical Analsys Cases I

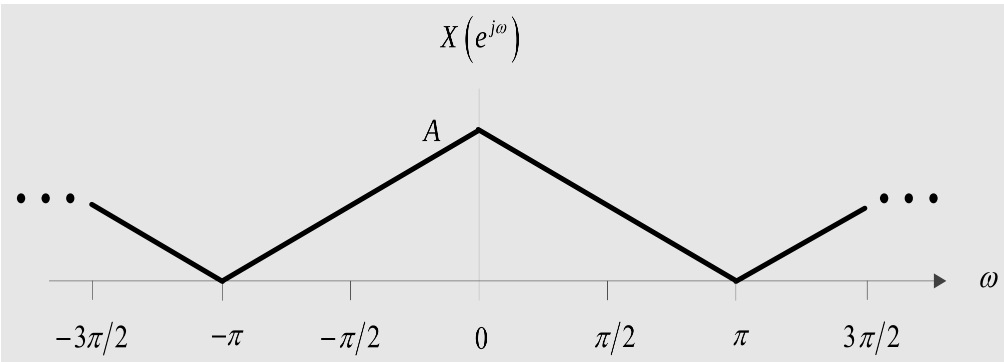

Figure 17(b). The DTFT of the system input of the LTI system of Mathmatical Analsys Cases I

The poly-phaser decomposition process of filter bank in figure 15(a) can be reorganized to be the filtering structure in figure 15(b), which requires only half the computational complexity of the filtering structure in figure 15(a).

Like the poly-phaser decomposition of a two-channel analysis filter bank processing (in figure 15(a) and figure 15(b)), the two-channel synthesis filter bank processing can be reconstructed by using the poly-phaser decomposition and the overall poly-phaser decomposition of a two-channel synthesis/analysis filter bank processing can be reorganized as shown in the following figure.

III. MATHMATCAL ANALYSIS CASE STUDY OF MULTI-RATE SIGNAL PROCESSING CONCEPT

A. *Mathmatical Analysis Cases I*

Consider the LTI system as shown in the following figure 17

First this section models the Discrete Time Fourier Transform (DTFT) of the signal $g_0[n]$ in the mathematical form of $H_0(e^{j\omega})$ and $X(e^{j\omega})$ for a general form of $H_0(e^{j\omega})$ and $X(e^{j\omega})$.

$$G_0(e^{j\omega}) = \tfrac{1}{2}\left(X(e^{j\omega})H_0(e^{j\omega}) + X(e^{j(\omega+\pi)})H_0(e^{j(\omega+\pi)})\right)$$

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Later, this section illustrates the Discrete Time Fourier Transform (DTFT) of the signal $x_0[n]$, $g_0[n]$ and $y_0[n]$ for $H_0(e^{j\omega}) = \begin{cases} 1 & |\omega| < \pi/2 \\ 0 & \pi/2 < |\omega| < \pi \end{cases}$ and $X(e^{j\omega})$ is shown in

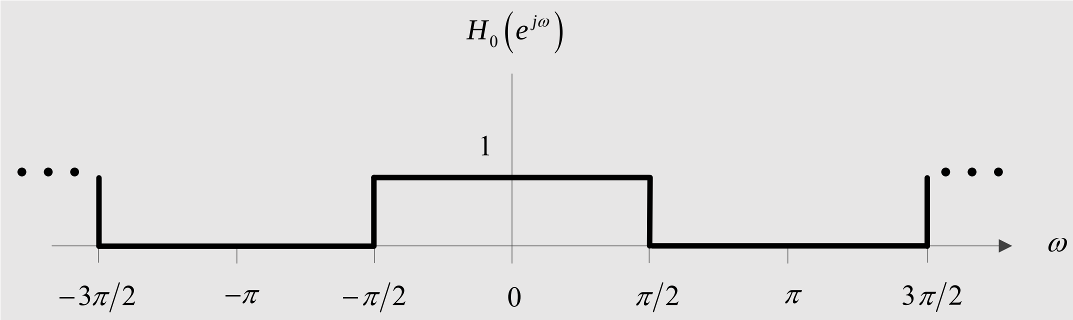

Figure 17(c) The DTFT of the system function $h_0[n]$ of the LTI system of Mathmatical Analsys Cases I

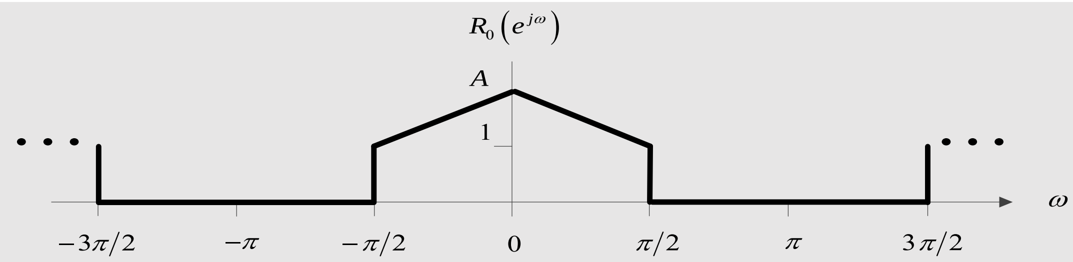

Figure 17(d) The DTFT of the $r_0[n]$ of Mathmatical Analsys Cases I

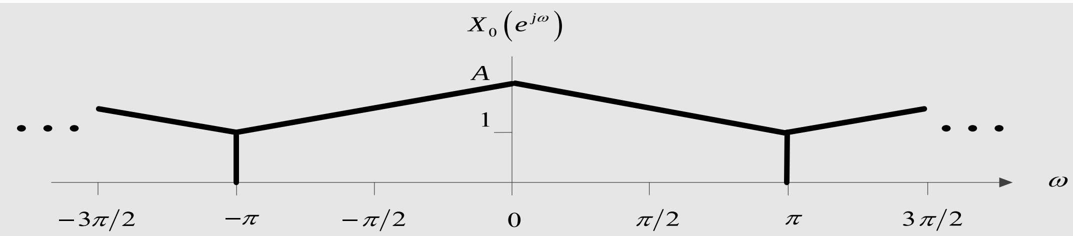

Figure 17(e) The DTFT of the down sampled signal $x_0[n]$ without aliasing of Mathmatical Analsys Cases I

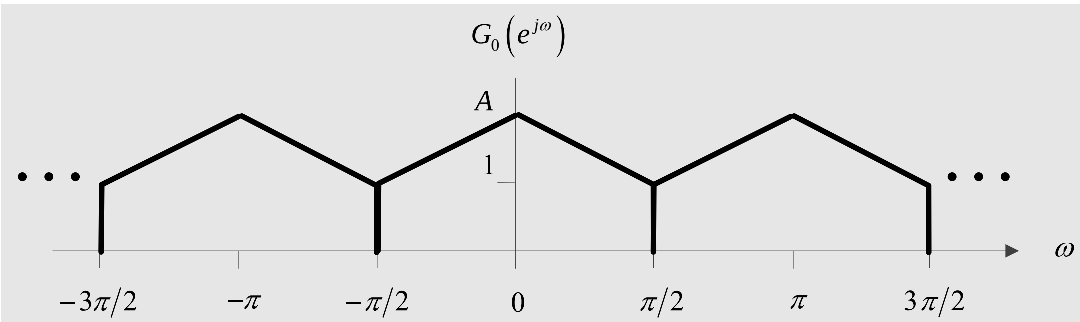

Figure 17(f) The DTFT of the down sampled signal $g_0[n]$ without aliasing of Mathmatical Analsys Cases I

Finally, this section models a set of conditions of $H_0(e^{j\omega})$, which is general and possible, and that will guarantee that $y[n]$ for any stable input $x[n]$ is proportional to $x[n-n_d]$.

$$Y_0\left(e^{j\omega}\right)=\tfrac{1}{2}H_0\left(e^{j\omega}\right)\left(X\left(e^{j\omega}\right)H_0\left(e^{j\omega}\right)+X\left(e^{j(\omega+\pi)}\right)H_0\left(e^{j(\omega+\pi)}\right)\right)$$

$$Y_1\left(e^{j\omega}\right)=\tfrac{1}{2}H_1\left(e^{j\omega}\right)\left(X\left(e^{j\omega}\right)H_1\left(e^{j\omega}\right)+X\left(e^{j(\omega+\pi)}\right)H_1\left(e^{j(\omega+\pi)}\right)\right)$$

$$Y\left(e^{j\omega}\right)=Y_0\left(e^{j\omega}\right)-Y_1\left(e^{j\omega}\right)$$

$$Y\left(e^{j\omega}\right)=\begin{cases}\tfrac{1}{2}X\left(e^{j\omega}\right)\left(H_0^2\left(e^{j\omega}\right)-H_1^2\left(e^{j\omega}\right)\right)\\+\tfrac{1}{2}X\left(e^{j(\omega+\pi)}\right)\left(H_0\left(e^{j\omega}\right)H_0\left(e^{j(\omega+\pi)}\right)-H_1\left(e^{j\omega}\right)H_1\left(e^{j(\omega+\pi)}\right)\right)\end{cases}$$

Due to $H_0\left(e^{j\omega}\right)H_0\left(e^{j(\omega+\pi)}\right)-H_1\left(e^{j\omega}\right)H_1\left(e^{j(\omega+\pi)}\right)=0$

Therefore

$$Y\left(e^{j\omega}\right)=\tfrac{1}{2}X\left(e^{j\omega}\right)\left(H_0^2\left(e^{j\omega}\right)-H_1^2\left(e^{j\omega}\right)\right)$$

From the above analysis, the aliasing mathematical terms traditionally cancel. $Y\left(e^{j\omega}\right)$ is directly dependent only on $X\left(e^{j\omega}\right)$ where $\left[H_0^2\left(e^{j\omega}\right)-H_1^2\left(e^{j\omega}\right)\right]$ is a constant.

IV. Conclusion

The primary propose of this paper is to highlight the mathematical model of the multi-rate signal processing concept of both down/up sampling multistate and poly-phase decompositions for digital systems and digital signals for future investigation and analysis. The significance of our work is routed in the fact that the multi-rate signal processing concept has been one of the most efficient and useful mathematical contexts in digital system applications, digital communication [14] and Digital Signal Processing (DSP) [1, 5, 13] during the last three and half decades. Hence, a great deal of mathematical modelling analyses and formulations were developed in this paper.

In addition, we employed the multi-rate signal processing concept in the scrutinized areas of image enhancement founded on SR (Super Resolution) concept [6, 7, 8] for enhancing the quality of signals in discrete-time sampling applications.

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