On the Convergence of NSGA-II Algorithm to a Single Pareto Optimal Solution with a Continuous and Quadratic Multi-objective Optimization Program

Raïmi Aboudou Essessinou, Guy Degla

Institute of Mathematics and Physical Sciences, University of Abomey-Calavi (UAC), Dangbo, Benin.
Email: raimi.essessinou@imsp-uac.org, gdegla@imsp-uac.org

Abstract **- In this paper, we test a fast elitist Non-Dominant Sorting Genetic Algorithm (NSGA-II) on a continuous quadratic multiobjective optimization problem. We use a big size of multiobjective programming for the Gross Domestic Product (GDP) quarterly disaggregation and we test different values of the NSGA-II parameters. We come to the conclusion that if the parameters are judiciously chosen, the NSGA-II algorithm converges to a single Pareto optimal solution in a regular and bounded set for the quadratic multiobjective optimization problem we used. It should be noted that elitism in the NSGA-II algorithm contributes to accelerating the rate of convergence and the overall performance of the genetic algorithm incorporated in it.**

Keywords - Quadratic Programming, Multiobjective Optimization, Genetic Algorithm, NSGA-II Algorithm, Pareto optimality, Quarterly disaggregation of Gross Domestic Product

I. INTRODUCTION

Several algorithms have been developed in the literature to solve multiobjective optimization problems, but most of these multiobjective optimization problems are NP-difficult problems [4] according to the size of the problem. Two types of approaches are adopted: non-Pareto approaches that seek to reduce the initial problem to one or more single-objective problems and Pareto approaches that do not transform the objectives of the problem and treat them simultaneously during resolution by seeking better compromise solutions [4].

A multiobjective programming problem is generally formulated as follows:

$$\min_{x\in U} F(x) = \left(f_1(x), f_2(x), \dots, f_Q(x)\right) \qquad \text{(MOPP)}$$

where $F:\mathbb{R}^N \longrightarrow \mathbb{R}^Q$ is the vector of objective functions and $U \subseteq \mathbb{R}^N$ the set of all the equality and inequality constraints.

A solution x is a vector of N decision variables: $x = (x_1, x_2, \dots x_N)$, $x \in \mathbb{R}^N$. Each decision variable x_i is framed by a lower bound x_{li}, and an upper bound x_{ui}. f_i represents the q^{th} objective function. Q is the number of objective functions and generally $Q \geq 2$.

Let's consider the vector function $F(x) = \left(f_1(x), \dots, f_Q(x)\right)$ and U the constrained space (space of feasible solutions) of the problem (MOPP). It should be noted $F(U)$ the value space of the objective functions. We define on $F(U)$ a partial relationship. Let K be any blunt cone such that $K \subset \mathbb{R}^Q$. Let's consider the binary relationship $\preccurlyeq_K$ indexed by K defined as follow:

$$a \preccurlyeq_K b \Leftrightarrow (b-a) \in K.$$

Since it is not possible to find a solution that simultaneously optimizes all objective functions in the case of a multiobjective program, the notion of dominance in the sense of Pareto is used.

(i)- Pareto-dominance: for two feasible decision vectors x et y, we say that x dominates y in the sense of Pareto and we note $(x, F(x)) \preccurlyeq_K (y, F(y))$, if and only if $\forall q \in \{1,2,3,\dots,Q\}$, we have $f_q(x) \leq f_q(y)$ and $\exists\, q_0 \in \{1,2,3,\dots,Q\}$ such that $f_{q_0}(x) < f_{q_0}(y)$.

(ii)- Pareto optimal solution: a solution $x \in \mathbb{R}^N$ is called Pareto optimal in $\mathbb{R}^N$ if and only if, there is no vector $y \in \mathbb{R}^N$ which dominates x.

(iii)- The Pareto optimal set is defined as the set of all Pareto optimal solutions.

(iv)- The Pareto optimal front is defined as the set of all objective function values corresponding to all solutions in the Pareto optimal set.

In general, it is difficult to have effective cooperation in the search for Pareto optimal solutions with a high number of objective functions. In fact, a multiobjective optimization program gives multiple Pareto optimal solutions in general. Thus, looking for a suitable algorithm for solving the quarterly disaggregation of GDP as a continuous quadratic multiobjective programming, our main idea is to apply the NSGA-II to prove if it seems promising to give a unique solution. This paper tries to check if a fast elitist multiobjective Non-Dominant Sorting Genetic Algorithm (NSGA-II) can converge towards a single Pareto optimal solution within the framework of the quarterly disaggregation of the GDP with a multiobjective optimization program.

II. A BRIEF OVERVIEW OF THE NSGA-II ALGORITHM

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The NSGA algorithm called "Non-dominant Sorting Genetic Algorithm" [9], [10], after several years of implementation, has become ineffective in ranking individuals [7] especially for large problems treated. To correct this deficiency, it elitism was introduced into the basic algorithm [6] in order to preserve the best solutions from generation to generation [7]. Thus, the improved version of the NSGA algorithm, called NSGA-II, classify the individuals into many levels [5], this classification does not require necessary the choice of sharing parameter [3]. The NSGA-II is illustrated by the pseudo code given in algorithm 1.

Algorithm 1 (pseudo-code NSGA-II) [8, 14]

```
While (total number of iterations not completed)
    Generation of the initial population
    Repeat
        While (Population is not classified) do
            Evaluation of all borders
            Normalization of areas of constraint violation
            Search for undominated individuals
            Replacement of individuals
        End while
        Selection
        Crossover
        Polynomial mutation
    Until (Criteria for shutdown achieved)
        Recombination of optimal Pareto solutions
End (while)
```

III. PRESENTATION OF THE TESTING PROBLEM

We use a problem of quarterly disaggregation of Gross Domestic Product (GDP). The GDP is decomposed into three sectors. Simulations are carried out using the databases of the National Institute of Statistics and Economic Analysis (INSAE). By grouping together all the elementary objective functions identified by the equation (7) for all branches, the quarterly disaggregation of GDP appears as a problem formulated in the form of multiobjective programming. Thus, the multiobjective programming temporal disaggregation model proposed in this research is as follows:

$$\min_X \{(f_1(X), f_2(X), f_3(X))\} \quad (1)$$

subject to

$$X = (X_1, X_2, X_3) \quad (2)$$

$$X_k = (X_{k,t})_t \ ; \ \forall t; \ k \quad (3)$$

$$-X_{k,t} \leq 0 \ ; \quad \forall \ t, k \quad (4)$$

$$\sum_{t=4y-3}^{4y} X_{k,t} - Z_{k,y} = 0 \ ; \ \forall y \ , \ k \quad (5)$$

$$\sum_{t=4y-3}^{4y} \frac{X_{k,t}}{I_{k,t}} \eta_{k,t} = \frac{Z_{k,y}}{\sum_{t=4y-3}^{4y} I_{k,t}} \ ; \ \forall \ k, y \quad (6)$$

Where

- T : the number of years for national accounts observed,
- $y \in \{1, 2, 3, \dots, T\}$, year generic index,
- $i \in \{1, 2, 3, 4\}$, quarterly index,
- $t \in \{1, 2, 3, \dots, 4T\}$: generic index of quarters on the T years' period obtained from i and y using an operator proposed in [8]:
- $k \in \{1, 2, 3\}$, generic branch index of the quarterly accounts nomenclature;
- Value of the annual account per branch: let $Z_{k,y}$ be the (known) value added of the branch account k , for the year y;
- quarterly indicator: let $I_{k,t}$ be the value of the branch indicator k, for the quarter $t = 1, 2, 3, \dots, 4T$;
- $I_k = (I_{k,t})_{t=1,2,\dots 4T}$: the vector of quarterly indicator related the branch k over the entire period;
- Inter-branch interaction: we note $\overline{W}_{j,k}$ the interaction of the branch j on the branch k, considered as the average share of the branch k demand of product coming from the branch j : $\overline{W}_{j,k} = 1 \ if \ j = k$ and $0 \leq \overline{W}_{j,k} < 1 \ if \ j \neq k$.
- Value added per branch: let $X_{k,t}$ be the value of national account for branch k at quarter $t = 1, 2, 3, \dots, 4T$
- The vector of national account value for branch k over the entire period is noted: $X_k = (X_{k,t})_{t=1,2\dots 4T}, X_k \in \mathbb{R}^{4T}$; $for \ k = 1, 2, 3$;
- The vector of quarterly national accounts value for all branches over the entire period is noted: $X - (X_1, X_2, X_3), X \in \mathbb{R}^{4T\times 3}$;
- Thus, the objective functions are given $\forall \ k$ by :

$$f_k(X_1, X_2, X_3) = \sum_{j=1}^{3} \sum_{t=2}^{4T} \overline{W}_{j,k} \left(\frac{X_{j,t}}{I_{j,t}} - \frac{X_{j,t-1}}{I_{j,t-1}} \right)^2 ; \quad (7)$$

As it can be seen, this problem is multiobjective quadratic programming.

Remark 1

The equation (1) is the main target multiobjective program.

The equation (2) means that X is a vector of quarterly national account values for all the branches over the entire period.

The equation (3) means that X_k is a vector of quarterly national account values for branch k over the entire period.

The equation (4) means that all the variables must have non negative values as the branches annual accounts are positive.

The constraint presented by equation (5) means that in each branch, the sum of the quarterly accounts estimated for the four quarters of a year is equal to the annual account (value added) of the branch for that year.

Denton's proportional method implicitly establishes from the observed annual benchmark indicator (BI) ratio a time series of estimation-indicator ratios (quarterly BI ratio) of quarterly estimates calibrated from the quarterly national accounts that is as smooth as possible and such that, in the case of flow series. Hence, for retrospective series ($y \in \{1, 2, 3, \dots, T\}$), the average of the quarterly BI ratio is equal to the annual BI ratio for each year y. so the equation (6) is to present the equilibrium relating to the weighted averages of BI ratios.

IV. OUR ALGORITHM PRESENTATION

The resolution algorithms of the traditional quarterly disaggregation methods are presented in [1], [2]. The simulation method for the model resolution is based on the NSGA-II algorithm developed in the literature for multiobjective optimization [3], [6], [9]

The NSGA-II adapted to the problem of quarterly disaggregation of GDP resulted in a fast elitist multiobjective programming algorithm. The pseudo code of the adapted NSGA-II algorithm is given in Algorithm 3.

In the NSGA-II algorithm, the initial population is generated from the bounded values (minimum and maximum) of the target variables. Since the problem is relatively of large size, to ensure rapid convergence of the algorithm towards Pareto optimal solutions, the solution domain has been narrowed using the annual data provided on account variables. Thus, the variables bounded values were generated from the annual values of the accounts by releasing the constraints (i) of the problem. The pseudo code of the limit value calculation procedure is given in algorithm 2.

Algorithm 2 Bounded_values (abX, ny)

```
BEGIN
*/Initialization of the annual accounts
For k = 1 : 3
For y = 1 : lyear
abX(k,y) ← annual values setting ;
end for
End for
*/Calculation of bounded values for quarterly accounts
For k =1 : 3
qbX_max (k) ← [ ] ;    */ empty box
qbX_min (k) ← [ ]  ;   */ empty box
For y = 1 : ny
qbXmax ← [ ] ;
qbXmin ← [ ] ;
 For t = 4y-3 : 4y
qbX_max (k,t) ← abX(k,y)/4+[SD (abX(k))]/(lyear-fyear+1) ;
qbX_min (k,t) ← abX(k,y)/4-[SD (abX(k))]/(lyear-fyear+1) ;
qbXmax ← [qbXmax qbX_max (k,t) ] ;
qbXmin ← [qbXmin  qbX_min (k,t) ] ;
t ← t+1 ;
End for
qbX_max (k) ← [qbX_max (k) qbXmax] ;
qbX_min (k) ← [qbX_max (k) qbXmin] ;
y ← y+1 ;
End for
qbX_M(k) ← qbX_max (k, 1:nq) ;
qbX_m(k) ← qbX_min (k, 1:nq) ;
k ← k+1;
End for
tXmax←[qbX_M(1) qbX_M(2)  qbX_M(3)];
tXmin ← [qbX_m(1)  qbX_m(2)  qbX_m(3)];
END
```

Finally, the pseudo code of the complete algorithm is given in algorithm 3 which is as follows:

Algorithm 3 (main pseudo code of MOPTD-NSGA-II)

```
BEGIN (main Algorithm)
*/Initialization time parameters
fyear     */ first year of available of annual accounts;
lyear     */ last year of available of annual accounts;
wyear     */ last year considered for interpolation;
*/ Calculation of all other associated parameters
wy ← (wyear - fyear +1) */ number of years for interpolation period;
wq ← 4*wy   */ number of quarters for interpolation period;
If lyear > wyear then
          ny← wy   */ number of years for interpolation;
          nq← wq  */ number of quarters for interpolation;
else
    ny ← (lyear – fyear +1);
    nq← 4*ny;
End if
*/ Algorithm 2 calling
Bounded_values_Calculation (abX, ny ) */ Execution of the Procedure
*/ Interpolation of accounts with quarterly indicators
Begin (model solution computation)
*/Initialization of the other NSGA-II input parameters
p1← number of constraints
V← 3×nq   */calculates the number of variables
Pop_size ← give the size of the population
run ← the number iteration
gen_max ← give the maximum number of generations
```

```
*/The Model function calling
MODEL FUNCTION
*/The Pareto optimal solution computation
        Begin (NSGA)
            Execution of the NSGA-II algorithm
        End (NSGA)
        */ Recovery of the optimal solution provided by NSGA
        If run==1
              qX ← [new_pop(:,1:V)] */ optimal solution
        Else
              qX ← [pareto_rank1(1:V)]  */ optimal solution
End if
End (solution computation)
END (main algorithm)
```

V. THE MAIN RESULTS OF THE PARETO OPTIMAL SOLUTION COMPUTATION

Then, several tests were performed on the key parameters of the algorithm in order to retain values that reduce the convergence time of the algorithm.

The NSGA-II algorithm uses the following key parameters as input parameters: population size (pop_size), iteration number (no_rum) and maximum number of generations (gen_max) beyond which the algorithm stops. The choice of the values of these parameters depends on the size of the problem. In the literature, problems are tested with the NSGA-II algorithm for minimum required values and set at 20 for pop_size and 5 for gen_max [3]. Sometimes, large values can be set: up to 200 for pop_size, 10 for no_rum and 1000 for gen_max in the case of problems with two objectives [6].

Based on these findings, several values were tested for the parameters pop_size as in [7], no_rum, and gen_max, in this research. These tests have been illustrated by different representations of the Pareto front. The Pareto front obtained for the main values of (pop_size, no_rum, gen_max) are presented in Figure 1 and Figure 2. The Pareto optimal points are represented in blue.

The analysis of the results shows that if no_run=1, the simulation gives several Pareto optimal points but the number of points increases with the population size (pop_size) and this regardless of the gen_max value, this situation is illustrated by the panels (a1) (b1), (c1) and (c2). It was used in the following pop_size=100 simulations, as adopted in [5], [10]. The Pareto boundaries presented in the panels (a2), (b2), (d1) and (d2) indicate that the algorithm converges to a single Pareto optimal point when no_run=100 for any gen_max value with pop_size=100, but the boundaries obtained with gen_max=25 and gen_max=100 have the same configuration; the points obtained for the different cases are all located in practically the same restricted space.

It should be noted that overall, the gen_max values set at 25 and 100 give Pareto front with the same characteristics; however, the runtime of the algorithm for displaying results is relatively longer for gen_max=100 than for gen_max=25. Thus, based on the situations presented above, the final simulations are performed with (pop_size, no_rum, gen_max) = (100, 10, 25).

VI. CONCLUSION

In this paper, we focus on the quarterly disaggregation of the GDP using an indirect approach. The quarterly disaggregation of GDP is formulated as a multiobjective programming model. Looking forward to finding a suitable algorithm to solve the problem, this paper provides a technical experience of NSAG-II algorithm by numerical testing of existence a single Pareto optimal solution. The tests are based on the parameters value which is suitable to ensure the singleness of the Pareto optimal solution found by the NSGA-II algorithm.

The multiobjective programming for the Gross Domestic Product (GDP) quarterly disaggregation appears as a big size problem. After testing different values of the NSGA-II parameters, we come to the conclusion that if the parameters are judiciously chosen i.e pop_size, ≥ 100, no_rum > 1 and gen_max > 2, the NSGA-II algorithm converges to a single Pareto optimal solution for the multiobjective optimization problem we used.

It should be noted that elitism in the NSGA-II algorithm contributes to accelerating the rate of convergence and the overall performance of the genetic algorithm incorporated in it. However, it should be noted that there is a random effect in the algorithm.

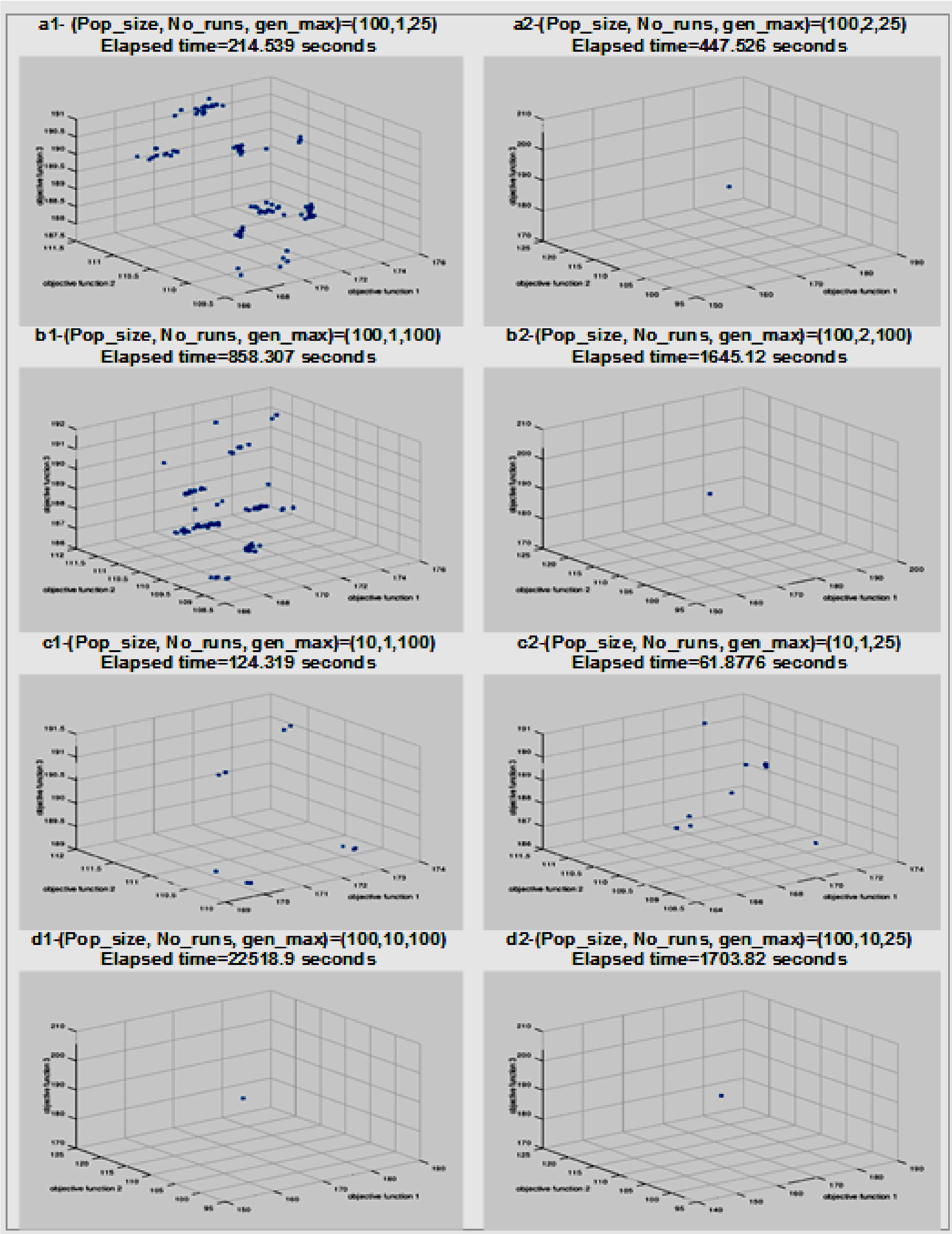

Figure 1. Figure 1. Optimal Pareto solutions for different simulation parameters

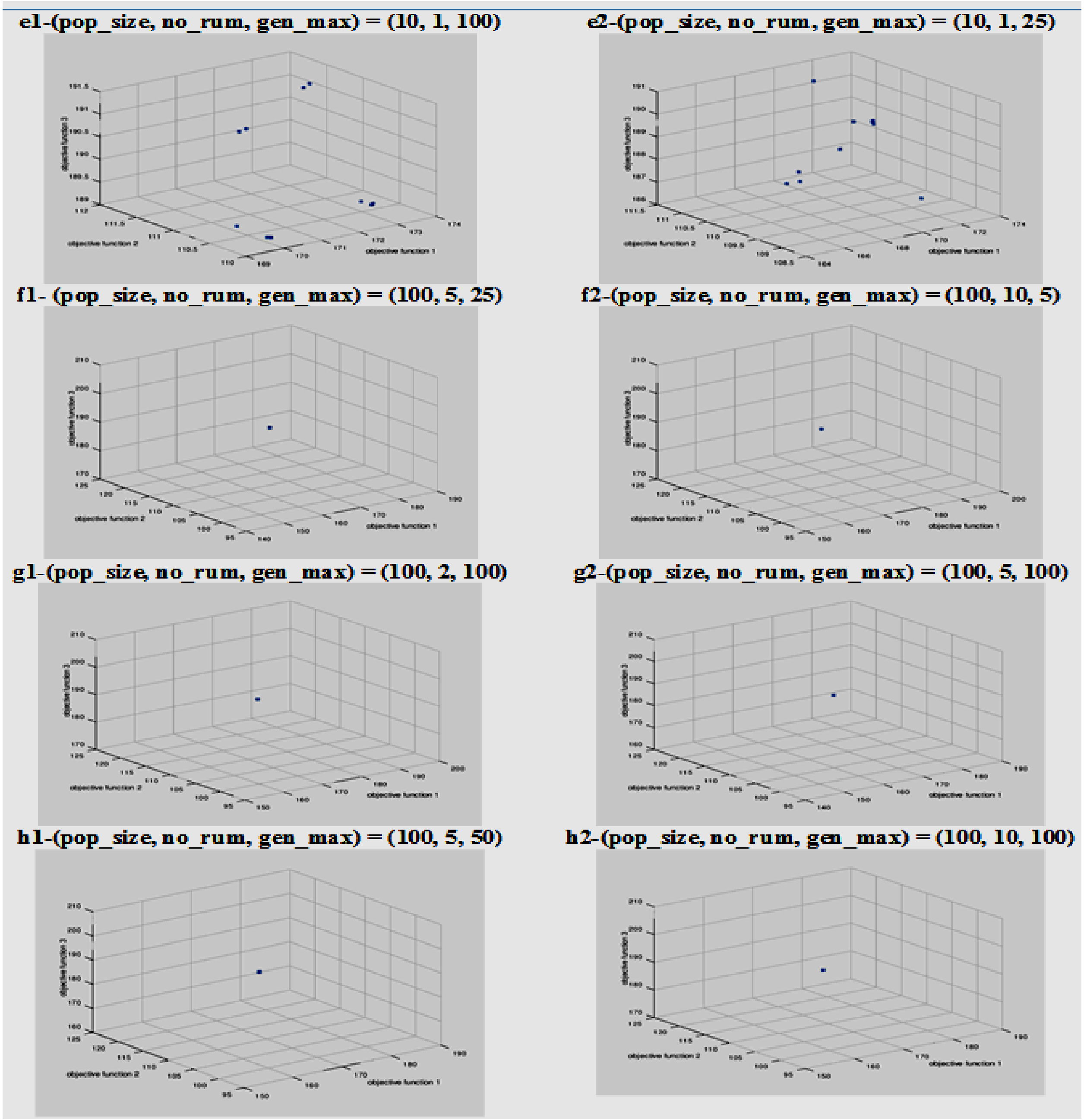

Figure 2. Various Optimal Pareto solutions for different simulation parameters

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REFERENCES

[1] Abdelwahed, T., Leila, H. : A Benchmarking Approach for Temporal Disaggregation of Economicc Time Series by Related Series. Working paper and studies, ISSN 1725-4825, ISBN 92-79-01306-8,

Luxembourg: Office for Official Publications of the European Communities, 5-10, 29-41 (2005). http://europa.eu.int.

[2] Ana, A., Enrique M. Q. : Software to perform temporal disaggregation of economic time series. Working paper and studies, ISSN 1725-4825, ISBN 92-894-8626-0, Luxembourg : Office for Official Publications of the European Communities , 3-40 (2005). http://europa.eu.int.

[3] Aravind, S. : A Fast Elitist Multiobjective Genetic Algorithm: NSGA-II. Mathlab source code (2012). http://www. mathworks. com/matlabcentral/fileexchange/10429-nsga-iia-multi-objective-optimization-algorithm/content

[4] Barichard, V. : Approches hybrides pour les problèmes multiobjectif. Thèse de Doctorat Num. d'ordre : 593, école doctorale d'Angers, Laboratoire d'Etude et de Recherche en Informatique d'Angers, Univesite d'Angers, 1, 7, 22, 32-38 (2003)

[5] Belaidia, I., Mohammedib K., Brachemi, B. : Mise en œuvre de l'Algorithme NSGA-II pour l'Optimisation Multi-Objectif des paramètres de coupe en Fraisage en bout. 19ème Congrès Français de Mécanique, Marseille, 24-28 août 2009, 3-5 (2009) https://www.researchgate.net/publication/260553237_Mise_en_oeuvre_de_l'Algorithme_NSGAII_pour_l'Optimisation_Multi-Objectif_des_parametres_de_coupe_en_Fraisage_en_bout

[6] Deb, K., Pratap, A., Agarwal, S., Meyarivan, T. : A Fast and Elitist Multiobjective Genetic Algorithm: NSGA-II. IEEE Trans on Evol. Comp, Vol. 6, No. 2, april 2002, 183 - 188 (2002), https://www.iitk.ac.in/kangal/Deb_NSGA-II.pdf

[7] Hafid, Z. : Représentation de solution en optimisation continue, multi-objectif et applications. Thèse de Doctorat cotutelle, Ecole Mohammadia d'Ingenieurs, LERMA- Rabat-Maroc, Institut National des sciences appliquées de Rouen, HAL Id: tel-00939980,30-31, 50-51 (2013), https://tel.archives-ouvertes.fr/tel-00939980/document, , 31 Jan (2014)

[8] Raïmi Aboudou Essessinou, Guy Degla and Babacar Mbaye Ndiaye: From Two to One Index Isomorphism in Optimization Program for Quarterly Disaggregation of Annual Times Series. Journal of Advances in Mathematics and Computer Science, 34(1&2): 1-15, 2019; Article no. JAMCS.51632, http://www.journaljamcs.com /index.php/JAMCS/article/view/30199

[9] Robin C. P., Kalyanmoy D., Maszatul M. M., Sanaz M., Rui W. : A Review of Hybrid Evolutionary Multiple Criteria Decision Making Methods. COIN Report Number 2014005, 1-3, 6 (2014)., https://www.egr.msu.edu/~kdeb/papers/c2014005.pdf

[10] Srinivas N., Kalyanmoy D. : Multiobjective optimization using Nondominated Sorting Genetic Algorithm. Journal of Evo. Comput. Vol. 2, No. 3, Pages 221-248, 7-12 (1994), https://web.njit.edu/~horacio/